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# **Optimal fiscal policy in the face of oil windfalls and other known unknowns**

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## Abstract

When a country discovers large new natural resource energy reserves, its citizens are gifted with possibility of financing a great economic transformation. Yet, often, the state which manages the transformation fails to deliver a sufficient high and well timed contribution to the well being of current and future citizens. Is it a lack of depth in financial markets or political impatience to spend that best explains the failure of energy revenues to achieve its potential? I answer this question with a model where both resource revenues and returns to the public sector's financial and physical assets and its liabilities are uncertain. Calibrating for Colombia, I find that when policymakers discount at 20%, 18pp below the social rate, the welfare cost is equivalent to about a quarter of consumption for ever, the net stock of assets is much smaller than it should be and discretionary expenditures are very sensitive to surprise resource revenues. If financial markets are sufficiently underdeveloped, we can generate welfare costs of the same magnitude. But imperfect financial markets cannot also explain why there are insufficient net effective assets as seems to be the case in Colombia, nor can they explain a heightened sensitivity to revenues. As political impatience is thus the main culprit, a continued emphasis on fiscal policy arrangements that directly address the temptation to spend too soon is warranted.

**Key words:** Energy resource, Fiscal policy, Financial development, Welfare

# 1 Introduction

When a country discovers new oil reserves, its citizens are gifted with a new source of revenue that though potentially capable of beneficial economic transformation is also dangerously volatile. The assets and liabilities of the state can be used to manage this unpredictability, but the returns to these instruments can themselves also be expected to be uncertain. How should fiscal policy be aligned to take best advantage of the windfall in the face of known unknowns?

History has shown that an abundance of tradable natural resources can often turn out to be destructive.<sup>1</sup> Recent empirical work has pinpointed excessive volatility as lying on the causal link between resource amplexness and low growth. van der Ploeg and Poelhekke (2009)'s estimations on a cross-country panel for 1970 to 2003 reveal that the ill effects of having natural resources are related to a greater unpredictability of per capita growth. Collier and Goderis (2008) find strong evidence from a dynamic panel estimates that high-rent non-agricultural commodity booms have only short-lived favourable effects on output; and the lower average growth rate of commodity-exporting economies is almost entirely due to a higher incidence of sharp slowdowns.

In principle, when faced with an uncertain windfall, the state should be able to shield its citizens by managing its gamut of assets and liabilities. Most of the earnings should be parked with an immediate reduction in debt or as an inflow into a wealth fund (van der Ploeg and Venables (2011)) and spent on private consumption at later dates. Investment in public capital should be timed such that the stock provides a steady stream of services to citizens. The fact that volatile and low growth has been a feature of many resource-rich countries might imply that there has not been enough precautionary saving in total, or of a sufficient efficiency.

In this paper I evaluate the implications of the different mechanisms by which volatility can come to lower growth in oil exporting countries. First there is the most simplistic possibility that the uncertainty in revenue is simply too great. Second, that due to political economy failures, fiscal decisions are discounted at a higher rate than is socially optimal. Then two final explanations relate to different aspects of financial frictions: that the returns on the instruments available to the governments that earn resources are simply too volatile and that asset returns are positively correlated with revenues — procyclical — such that the portfolio cannot diversify the income risk.

Natural resource revenues, such as that from oil, are notoriously unpredictable. In large part this is because of the enormous volatility in the world oil price. As evidence, it is sufficient to marvel at the roller coaster ride of the WTI oil price in recent times, which rose from 20 dollars a barrel in 2004 to 147 dollars in July 2008, fell to 33 dollars in January 2009 and picked up again to 100 dollars in 2010. While there has been much emphasis on prices, it is noteworthy that the costs of extracting oil can also be difficult to forecast. Extraction cost uncertainty is likely to play an ever greater role in oil, because as traditional reserves

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<sup>1</sup>See, for example, surveys by Frankel (2010) and van der Ploeg (2011) and the case studies in Collier and Venables (2011).

are exhausted, the resource is being extracted in unfamiliar contexts: the deep sea, in oil sands or from shale.

Yet, there are historical examples of countries that have successfully managed to extract large net benefits from the volatile rents of natural resources. By comparing experiences, many have argued that political economy failures are responsible for poor outcomes.<sup>2</sup> Nearly all of the empirical papers that identify a significant negative effect of natural resource abundance, directly or through volatility, onto growth also find that this is alleviated by some measure of the quality of policy institutions.<sup>3</sup> Tornell and Lane (1999) suggest that if fiscal discipline is weak, interest groups may scramble to secure expenditure commitments. Alesina, Campante, and Tabellini (2008) show that suboptimally-timed procyclical spending may be the outcome if voters do not know how much of the oil rents are being appropriated by a corrupt state. A straightforward interpretation is that the rate of discount applied to fiscal policy decisions is higher than the social rate, and thus that future consequences are suboptimally disregarded.<sup>4</sup>

A third explanation is to do with limits to financial markets. One aspect is excess volatility in the returns on the government's investment opportunities. This is relevant because many resource exporting countries are developing countries, and one of the endemic features of a lower level of development is a paucity of efficient assets that collectively offer an adequate combination of risk and return. The lack of depth in financial markets has been recognised as a constraint to growth in the development literature and should apply to the government as much as to private citizens. Consistently, van der Ploeg and Poelhekke (2009) estimated that ill effects of volatility depend on the extent of underdevelopment of domestic financial markets. Broner and Rigobon (2005) estimate that the unconditional standard deviation of capital flows into less developed nations is on average eighty percent times larger than to developed counterparts partly because disturbances to financial market conditions for these countries tend to be more persistent.

Most of the literature on the fiscal management of natural resources has restricted attention to safe financial assets such as reserves. Debt is occasionally incorporated (often as negative reserves — by assuming that the rate of repayment is certain). But public capital is

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<sup>2</sup>van der Ploeg (2011) analyses and surveys the accumulated understanding political failures in resource abundant economies.

<sup>3</sup>Examples are Collier and Goderis (2008), Arezki and Bückner (2010) or Arezki and Bückner (2012). Frankel (2010) in reviewing the Chilean case suggest that one important aspect is that the forecasts on which fiscal policy are predicated are determined by an independent body.

<sup>4</sup>A different prediction of these models — in particular Alesina, Campante, and Tabellini (2008) — is that the greater noise in fiscal accounts induced by natural resource dependence could weaken monitoring. Then another route is that natural revenues raise the likelihood of future expenditure claims whose timing is enforced on the state. While both of these channels by which volatility can lower growth could in principle be analysed in my framework, I prefer to leave them for future work and focus on the implication of a suboptimally high discount rate. Then there are reasons why spending oil revenue immediately could be optimal. Segal and Sen (2011) argue that where there are many in absolute poverty, a large part of oil revenues should — optimally — be spent as soon as the revenue accrues to help those among the local population that are in a poverty trap, assuming that it is public expenditure that is needed to help push the poor out of that state and that elimination of the poverty trap is a bottleneck preventing further progress on the government's objectives. I don't propose to model this intriguing dilemma here either.

nearly always disregarded. This is presumably because returns to public investment are risky. Gupta, Kangur, Papageorgiou, and Wane (2011) find that investment is highly inefficient in creating capital stock on average across all emerging market countries while Bhattacharyya and Collier (2011) show that this seems to be especially true of natural resource exporting reporting countries.

Indeed, over the recent two decades, resource revenues have been parked in foreign reserves with low risk (at least in dollar terms) and debt liabilities have also lowered. While this is prudent, it is also important to recognize that these instruments imply a very low real rate of social return, quite possibly insufficient to compensate even for the elimination of risk.<sup>5</sup>

The reason why the omission of public capital is glaring is that, despite its riskiness, it makes a vital contribution to the welfare of its citizens. Macroeconomists often discuss the countercyclical role of public investment, as an injection into private domestic incomes at the time of making the expense, independent of its economic function. The controversy generated by this short-term boost masks that an effective public capital stock also yields utility to private citizens in the form of stream of future services, just as a consumer durable.<sup>6</sup> As these dividends only occur after a long and uncertain horizon following investment, it is not realistic to think of public investment as being manipulated systematically to counteract previously unanticipated shocks to resource returns. But portfolio theory tells us that as long the service flow injection into private utility is uncorrelated with other shocks such as resource rents, there could still be social benefit from effective public investment (see Collier and Venables (2011) page 23). Hence large level of reserves or low level of debt are not necessarily consistent with an optimal level of state assets; we should also consider the effective public capital stock. To match this against reality, there should be some explanation of why the effective quantity of the capital stock could be suboptimal.

A related aspect is that rather than being volatile, the returns on possible investments are correlated with the state's income risk, the natural resource revenue. Standard finance theory argues that under these circumstances, the undiversifiable component of risk in the government's portfolio is greater and its potency as a cushion diluted. Many authors have found that the lending terms on developing countries' external liabilities are more lax when GDP is strong,<sup>7</sup> and that the appetite of foreign investors tends to be positively related to the terms of trade or export prices (Reinhart and Reinhart (2008)). In fact, capital inflows are increasingly associated with FDI into the commodity producing sector.<sup>8</sup>

Not only foreign borrowing terms, but also *domestic* rates wax and wane with export

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<sup>5</sup>See Bernstein, Lerner, and Schoar (2009) for evidence of and explanations for low returns from sovereign wealth funds.

<sup>6</sup>Another argument in favour of including public capital that I do not include here is that many resource exporting countries are capital constrained (van der Ploeg and Venables (2011) and van der Ploeg and van den Bremer (2012)).

<sup>7</sup>See Kaminsky, Reinhart, and Végh (2004), for example, for evidence.

<sup>8</sup>UNCTAD (2007) report that 'in 2005, the 10 largest FDI recipient countries in Africa were rich in oil or metal minerals; and in Latin America, most economies with significant natural resources saw increases in FDI in primary industries.'



prices. This is most obviously true in those commodity-exporting countries with fixed exchange rates. But even in flexible exchange rate regimes, real domestic rates could become procyclical because of what is called a fear of floating. This is due to a combination of policy bias and the inherent procyclicality of banking sector. If capital inflows and buoyant resource revenues stimulate a greater inflow of deposits into the banking sector, domestic banks will most likely offer credit at lower rates. As bank credit is typically skewed to the non-tradable sector, the real exchange rate, being the price of non-tradables to tradables, appreciates strongly. This real appreciation also makes policymakers less able (or willing) to tighten domestic currency rates.<sup>9</sup> The fear of floating explanation is consistent with another regularity about when a dependence of natural resources implies low growth, that the real exchange is correlated with commodity price (Papyrakis and Gerlagh (2004)).<sup>10</sup>

Procyclicality has been enough of a concern to motivate proposals to index debt to commodity prices, such that debt service rises when the commodity price is high and falls when it is low (Frankel (2003) and Caballero (2002)) or to index GDP debt (Isakova, Plekhanov, and Zettelmeyer (2012) for example). There are also arguments that a greater countercyclicality in the real policy rate should be induced through targeting a commodity standard such as the price of exports in local currency or producer prices (Frankel (2010)). And countercyclical debt rates may also be a crucial feature of macroprudential policies in resource-endowed economies: Turner (2011) presents the case for using long-term debt rates as a financial stability policy instrument to lean against this wind. Therefore, this paper also represents an indirect test about the gamut of complementary policies which make the real return on government buffer assets covary positively with the revenue.

To estimate the consequences of these mechanisms, I build a model where the government receives an uncertain exogenous revenue stream, an important part of which is down to a natural resource. It can invest in an aggregate net asset, but the returns on those assets are themselves uncertain, and may be correlated with revenue. There is also the possibility of unalterable exogenous expenditure commitments. From this, the government has to extract a stream of net expenditure which it transfers to private consumers. The government can be more impatient than is socially optimal. I derive the welfare costs of these different sources of uncertainty and political impatience. I calibrate this to match modern day Colombia, a good case study because oil revenues will play an ever greater role in determining the country's economic future.

As a by product, I implement an important technical innovation in modelling the natural resource problem. I apply Coeurdacier, Rey, and Winant (2011)'s risky steady state concept to determine the optimal level of net assets. The risky steady state is defined intuitively where all variables are constant, or grow at a constant rate, while uncertainty is expected in the

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<sup>9</sup>See Hausmann and Rigobon (2003) for a formal model of this mechanism in commodity exporting countries and and Mahadeva and Pineda (2009) for sectoral evidence from Colombia.

<sup>10</sup>Where the central bank is targeting inflation with a flexible exchange rate, the real cost of domestic debt service is linked to the expected path of short policy rates (and various risk and term premia). It might matter than in a commodity price boom, the price of imported food and raw materials tend to fall as the exchange rate appreciates. However the standard practice is to ignore the temporary, first round effects of these changes and react only to implications for forecasted inflation.

future. This is the unconditional mean of the ergodic distribution of variables and is related to the stochastic steady state or long-run invariant distribution in optimal growth models. There is an explicit recognition that natural resource revenues, expenditure commitments, debt costs and returns to public capital can be uncertain. Fittingly, it permits a welfare analysis of the profound consequences of natural resource abundance, as volatility affects the steady state and not just the dynamics.

The bulk of the literature on fiscal policy for resource rich countries uses either a perfect foresight or a log-linear solution. A famous example is the Hartwick rule (Hartwick (1977)) which prescribes a path where fiscal saving is exactly equal to current resource extraction revenue with the objective of keeping consumption constant at some unspecified level. Certainty equivalence excludes the prudential motivation to limit net borrowing or hold assets. Thus net debt is indeterminate in these models and has to be either left indeterminate or imposed. The first option implies the unpalatable property that where debt settles depends on the initial level and the history of *temporary* shocks (Schmitt-Grohé and Uribe (2003)). The second leaves a crucial part of the problem unexplained.

The importance of precautionary saving for the fiscal problem of resource-producing countries has been tackled by for example Bems and de Carvalho Filho (2011), Arrau and Claessens (1992), van der Ploeg (2010) and recently van der Ploeg and van den Bremer (2012).<sup>11</sup> None of these papers allow for welfare comparisons as I do, however.<sup>12</sup>

In the next section (Section 2) the concept of a risky steady state solution is explained, as the original paper by Coeurdacier, Rey, and Winant (2011) provided only some of the details. In Section 3, I go on to describe the model of the fiscal problem and provide some intuition as to what we might expect from a numerical solution. Section 6 explains how I measure welfare. In Section 7, the calibrations for Colombia are presented and the baseline solution of the model is discussed. Sections 8, 9, 10 and 11 explore the effects of greater revenue uncertainty, asset return uncertainty, political impatience and pro- versus counter-cyclicality respectively. Section 12 compares these competing causes. Section 13 concludes.

## 2 A general formulation of the risky steady state problem

In this section, I describe a general formulation of Coeurdacier, Rey, and Winant (2011)'s analysis of a risky steady state as their paper did not go into much detail. This section can

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<sup>11</sup>Economists working on parallel problems have long since recognised the importance of non-certainty equivalent rational expectations and precautionary buffer stocks. See Deaton and Laroque (1996) or even the seminal contribution of Gustafson (1958) in commodities and Carroll (2004) in the wider macroeconomics literature. Samuelson (1971) argued that where speculation matters,

the minimal model for understanding market behavior must involve stochastic processes. ...  
The resulting stationary time series possesses an ergodic state.

<sup>12</sup>van der Ploeg (2010) held interest rates fixed while van der Ploeg and van den Bremer (2012) estimate the effect of return uncertainty but over two periods according to a uniform distribution.

be skipped by a reader uninterested in the justification for this method. The method belongs to the class of increasingly popular perturbation methods (Judd (1996)) but only that the steady state is stochastic.

## 2.1 Set up of the model

The model is described by  $N_y$  nonlinear equations in  $N_y$  endogenous variables:

$$\mathbb{E}_t [\mathbf{f}(\mathbf{y}_{t+1}^+, \mathbf{y}_t^+, \mathbf{y}_t, \mathbf{y}_t^-, \mathbf{y}_{t-1}^-, \mathbf{u}_{t+1}, \mathbf{u}_t)] = \mathbf{0}. \quad (1)$$

Here  $\mathbf{y}_t$  is the vector of  $N - N_y^+ - N_y^-$  endogenous variables appearing only as current values,  $\mathbf{y}_{t+1}^+$  are  $N_y^+$  variables of  $\mathbf{y}_t$  that appear with a lead and possibly in the current period but with no lag.  $\mathbf{y}_{t-1}^-$  are the  $N_y^-$  variables that appear with a lag and possibly in the current period but with no lead. By creating new variables we can eliminate variables that enter with both a lead and a lag.<sup>13</sup>  $\mathbf{u}_t$  are  $N_u$  exogenous variables which follow

$$\mathbf{u}_t = \mathbf{h}(\mathbf{u}_{t-1}, \epsilon_t) \quad (2)$$

where  $\epsilon_t \underline{d} N(\mathbf{0}, \Sigma)$ . Among the endogenous variables there could be Lagrange multipliers on constraints. Terminal conditions dictate that these variables converge to risky steady state. There are initial conditions for the predetermined or state variables,  $[\mathbf{y}_{t-1}^-, \mathbf{u}_t]$ .

We assume that a generalised version of the implicit function theorem holds (Edwards (1994), Theorem 3.4) such that we can describe all other variables as a unique function of known values of state variables and the derivative of this function can be calculated by implicit differentiation of 1.<sup>14</sup> For convenience we define  $\mathbf{z}_{t+1}$  as the vector of all  $N$  variables,  $\mathbf{v}_{t+1}^+$  as the vector of  $N_y^+ + N_u$  expected future variables, and  $\mathbf{v}_t^-$  as the  $N_y^- + N_u$  state variables:

$$\mathbf{z}_{t+1} \equiv [\mathbf{y}_{t+1}^+, \mathbf{y}_t^+, \mathbf{y}_t, \mathbf{y}_t^-, \mathbf{y}_{t-1}^-, \mathbf{u}_{t+1}, \mathbf{u}_t] \quad (3)$$

$$\mathbf{v}_{t+1}^+ \equiv [\mathbf{y}_{t+1}^+, \mathbf{u}_{t+1}] \quad (4)$$

$$\mathbf{v}_t^- \equiv [\mathbf{y}_{t-1}^-, \mathbf{u}_t]. \quad (5)$$

A necessary condition for the risky steady state, as with the deterministic steady state, is an absence of dynamics in detrended variables, or  $\bar{\mathbf{z}} = \mathbf{z}_{s+1} = \mathbf{z}_s$ . A deterministic steady state could be solved by imposing this condition to the system 1 directly. A risky steady state has to be derived jointly with the dynamic rational expectations solution.

## 2.2 The solution of the risky steady state

We begin by considering a second-order approximation to the  $i^{th}$  function ( $i = 1 \dots N_y$ ) on the righthandside of equation 1 about the expected future variables  $\mathbf{v}_{t+1}^+$ :

$$\mathbb{E}_t [f_i(\mathbf{z}_{t+1})] \approx \Phi_i(\mathbb{E}_t[\mathbf{z}_{t+1}]) \quad (6)$$

<sup>13</sup>Thus  $\mathbb{E}_t y_{t+1} - \alpha y_t + (1 - \alpha)y_{t-1} + e_{t+1} = 0$  can be written as  $\mathbb{E}_t y_{t+1} - \alpha y_t + (1 - \alpha)z_t + e_{t+1} = 0$  and  $z_t = y_{t-1}$ .

<sup>14</sup>The regularity conditions relate to the differentiability of the utility function and constraints.

where

$$\begin{aligned}
\Phi_i(\mathbb{E}_t[\mathbf{z}_{t+1}]) &\equiv f_i(\mathbb{E}_t[\mathbf{z}_{t+1}]) \\
&+ \sum_j \frac{df_i}{dv_{j,t+1}^+}(\mathbb{E}_t[\mathbf{z}_{t+1}])\mathbb{E}_t[(\mathbf{v}_{i,t+1} - \mathbb{E}_t[\mathbf{v}_{i,t+1}])] \\
&+ \frac{1}{2} \sum_j \sum_k \frac{d^2 f_i}{dv_{j,t+1}^+ dv_{k,t+1}^+}(\mathbb{E}_t[\mathbf{z}_{t+1}]) \\
&\times \mathbb{E}_t[(\mathbf{v}_{j,t+1} - \mathbb{E}_t[\mathbf{v}_{j,t+1}])(\mathbf{v}_{k,t+1} - \mathbb{E}_t[\mathbf{v}_{k,t+1}])^T]. \tag{7}
\end{aligned}$$

As  $\mathbb{E}_t[(\mathbf{v}_{i,t+1} - \mathbb{E}_t[\mathbf{v}_{i,t+1}])] = 0$ , equation 7 can be simplified as:

$$\begin{aligned}
\Phi_i(\mathbb{E}_t[\mathbf{z}_{t+1}]) &= f_i(\mathbb{E}_t[\mathbf{z}_{t+1}]) \\
&+ \frac{1}{2} \sum_j \sum_k \frac{d^2 f_i}{dv_{j,t+1}^+ dv_{k,t+1}^+}(\mathbb{E}_t[\mathbf{z}_{t+1}]) \\
&\times \mathbb{E}_t[(\mathbf{v}_{j,t+1} - \mathbb{E}_t[\mathbf{v}_{j,t+1}])(\mathbf{v}_{k,t+1} - \mathbb{E}_t[\mathbf{v}_{k,t+1}])^T]. \tag{8}
\end{aligned}$$

In the risky steady state, equation 8 becomes:

$$f_i(\bar{\mathbf{z}}) + \frac{1}{2} \sum_j \sum_k \frac{d^2 f_i}{dv_{j,t+1}^+ dv_{k,t+1}^+}(\bar{\mathbf{z}}) \text{Cov}_{t,t+1,j+,k+}(\bar{\mathbf{z}}) = \mathbf{0}. \tag{9}$$

These  $N_y$  equations are insufficient to yield a solution as there are more than  $N_y$  unknowns comprising the  $N_y$  risky steady state values  $\bar{\mathbf{z}}$  and also the  $(N_y^+) \times (N_y^+ + N_u)$  unknown conditional covariances.

In a next step, the dynamic rational expectations solution to the risky steady state system is assumed to be of the linear form

$$[\mathbf{y}_t^+, \mathbf{y}_t, \mathbf{y}_t^-] = [\bar{\mathbf{y}}^+, \bar{\mathbf{y}}, \bar{\mathbf{y}}^-] + \mathbf{G}([\mathbf{y}_{t-1}^-, \mathbf{u}_t] - [\bar{\mathbf{y}}^-, \bar{\mathbf{u}}]). \tag{10}$$

Note that the  $N_y$  posited expectational rules have  $N_y + N_y \times (N_y^- + N_u)$  unknowns: the matrix  $\mathbf{G}$  has  $N_y \times (N_y^- + N_u)$  elements and there are  $N_y$  risky steady state values  $[\bar{\mathbf{y}}^+, \bar{\mathbf{y}}, \bar{\mathbf{y}}^-]$ .

10 determine the unknown conditional covariances terms in equations 9 as functions of  $\mathbf{G}$ , the risky steady-state values and the conditional covariance matrix of the exogenous series (from equation 2):

$$\text{Cov}_{t,t+1,y^+,y^+}(\bar{\mathbf{z}}) = [\mathbf{G}^{+u}]^T \Sigma \mathbf{G}^{+u} \tag{11}$$

and

$$\text{Cov}_{t,t+1,y^+,u^+}(\bar{\mathbf{z}}) = [\mathbf{G}^{+u}]^T \Sigma \tag{12}$$

where  $\mathbf{G}^{+u}$  is the  $N_y^+ \times N_u$  partition of  $\mathbf{G}$  that reflects the influence of  $\mathbf{u}_t$  on  $\mathbf{y}_t^+$ .

$\Phi_i(\mathbb{E}_t[\mathbf{z}_{t+1}])$  in 8 is approximately a constant function — it is approximately equal to zero for any possible values of  $\mathbf{v}_t^-$ . Thus, given that a generalised version of the implicit function

theorem holds, it must be the case that the first derivatives of  $\Phi_i$  with respect to  $\mathbf{v}_t^-$  are also approximately equal to zero. Formally,

$$\Phi_{i,v_t^-,1}(\bar{\mathbf{z}}) \approx \mathbf{0}, \text{ for } i = 1 \dots N_y \quad (13)$$

where  $\Phi_{i,v_t^-,1}$  denotes the total derivative vector of  $\Phi_i(\mathbb{E}_t[\mathbf{z}_{t+1}])$  with respect to  $\mathbf{v}_t^-$ , evaluated at the risky steady state.

The  $N_y$  equations 9, the  $N_y \times (N_y^- + N_u)$  equations 13 and the  $(N_y^+) \times (N_y^+ + N_u)$  covariance terms in equations 11 and 12 can now be combined to solve the system for the unknowns, the risky steady state values ( $[\bar{\mathbf{y}}^+, \bar{\mathbf{y}}, \bar{\mathbf{y}}^-]$ ), the covariances, and the dynamic solution,  $\mathbf{G}$ , by numerical methods. In this way, the dynamic solution and risky steady state values are obtained together.

The task of solving this system might seem overwhelming given the number of solution values we will be searching for numerically. However, as state transitional equations are usually conditionally linear, the system can often be solved recursively and reduced to a smaller system in the risky steady state values of the state variables and the coefficients of their expectational rules.

As a first step, the  $(N_y^+) \times (N_y^+ + N_u)$  conditional covariances in equations 11 can be solved in terms of the known parameters defining the exogenous variables as well as the unknown risky steady-state values of the  $N_y^-$  endogenous state variables and the  $N_y^- \times (N_y^- + N_u)$  unknown coefficients in equation 12. Hopefully this reduction is analytical. Substituting for these terms into 11 gives a new system with unknown covariance terms eliminated.

Next we can reduce the system further to  $N_y^-$  equations in terms of the exogenous variables, the known parameters defining the exogenous variables as well as the  $N_y^-$  endogenous state variables and the  $N_y^- \times (N_y^- + N_u)$  unknown coefficients in equation 12.

Differentiating this new system in terms of the eight state values at time  $t$  and then evaluating at the risky steady state values gives  $(N_y^- + N_u)$  equations.

Taking the new system of  $N_y^-$  equations at the risky steady state values gives a further  $N_y^-$  equations.

Together these  $2N_y^- + N_u$  equations can be used to solve for the  $(2N_y^- + N_u)$  unknown values numerically, constituting the core of our solution.

### 3 Optimal fiscal rules in the face of risk

Having established the general solution method, I now turn to the specific problem of the policy management of uncertain natural resources.

A government typically has many different types of items on its current balance sheet. A first distinction is between those which are reproducible and those which are not. Reproducible assets earn a rate of return, while reproducible liabilities such as debt require interest payment. A full list of the assets of the state would be public sector physical capital, domestic financial investments, foreign reserves and wealth funds. Reproducible liabilities would be sovereign debt and bank loans. In what follows, I aggregate these into a net asset, allowing for imperfect substitutability.

As for non-reproducible expenditure and revenue streams, these are further distinguished into those which can be adjusted flexibly and those which cannot. The former are called discretionary items, the latter are non-discretionary. A related distinction is that underlying the *structural budget*, a popular decomposition which removes cyclical and other elements that are not controlled by the government, or at least those which respond passively to the environment. Accordingly, non-discretionary revenues (NDR) include the royalties from natural resource windfalls. Within non-discretionary expenditures (NDE), one could include pension liabilities, compensation for conflict, the resolution of bad banks or contributions to a disaster fund or more generally, the large notoriously rigid component of total government expenditure. Here finite natural resources are not depletable capital and the simultaneous decision of how much to extract is not modelled. The right to raise tax revenue could also be conceived as a capital stock whose capacity requires investment and yields a return. But here tax incomes are from a non-reproducible source.

Formally, the fiscal authority maximises the following additive, time-separable, infinite horizon welfare function:

$$\mathbb{E}_t U(C_t, \dots, C_\infty) = \mathbb{E}_t L_t \sum_{s=t}^{\infty} \beta^{s-t} \frac{L_s}{L_t} \frac{\left(\frac{C_s}{L_s}\right)^{1-\gamma} - 1}{1-\gamma} \quad (14)$$

with  $\gamma > 0$ . Here  $C_t$  is net discretionary government expenditure on all its citizens and  $L_t$  is the population at time  $t$ .  $\beta$  is the government's rate of discount which is assumed to be greater than the social rate of discount ( $\hat{\beta}$ ). The government is supposed to care about future generations weighted by their population size. There are three other important assumptions inherent in these preferences. First, that privately purchased consumption is additive in social welfare to discretionary expenditure. Second, that it is *net* public expenditure that matters; i.e. taxes do not matter independently. Third, that non-discretionary expenditures and revenues of the government are also additive in utility to the discretionary component.

The budget constraint of the state is

$$C_t = r_t \left( \frac{W_{t-1}}{\Lambda_t L_t \tilde{w}} \right)^{-\delta} W_{t-1} - W_t + X_{1,t} - X_{2,t}. \quad (15)$$

For the sake of argument, non-resource GDP grows at an exogenous rate  $\tau = \frac{\Lambda_t L_t}{\Lambda_{t-1} L_{t-1}}$  and  $\Lambda_0 = 1$ .

To finance net discretionary expenditures ( $C_t$ ), the government receives a non-discretionary incomes in real dollar terms of  $X_{1,t}$ , faces committed spending plans of  $X_{2,t}$  and can save in net an amount equal to  $W_t$  which earns a real gross rate of return in domestic currency. The rate of return on net assets is the product of two influences: an exogenous rate,  $r_t$ , as well as an endogenous component  $\left(\frac{W_{t-1}}{\Lambda_t L_t \tilde{w}}\right)^{-\delta}$ . The latter depends on the size of the net asset position,  $W_t$ , relative to a benchmark,  $\tilde{w}\tau^t$  and the parameter  $\delta$  measures the sensitivity of real return to this imbalance. As I shall demonstrate later, the endogenous term can be interpreted as the outcome of the opposing influences of a declining marginal product of capital and the importance of a net asset cushion for production. I show that there are good

reasons to argue that the first effect dominates such that  $\delta$  should be small and positive, though it should be closer to zero in economies with more financial frictions. This term is not required for there to be an endogenous equilibrium level of net assets, but it does mean that, plausibly, governments might respond to lower returns on net assets by investing less in efficient public sector capital. This exacerbates the consequences of procyclical debt rates and interferes with the functioning of a countercyclical rate policy.

The budget constraint (15) can be written as

$$c_t = x_{1,t} - x_{2,t} + \frac{w_{t-1}}{\tau} \left( \frac{w_{t-1}}{\tau \tilde{w}} \right)^{-\delta} r_t - w_t \quad (16)$$

where lower case denotes model units; i.e. that all real volume variables have been divided by time  $t$  technical progress and population size which together grow at a constant rate; i.e  $y_s \equiv \frac{Y_s}{\Lambda_t} \tau^{t-s}$  for any real volume variable,  $y_s$ .<sup>15</sup> Thus the variables in equation 16 can be considered to be stationary.

At the risky steady state, equation 16 becomes

$$\bar{c} = \bar{x}_1 - \bar{x}_2 + \frac{\bar{w}}{\tau} \left( \frac{\bar{w}}{\tau \bar{w}} \right)^{-\delta} \bar{r} - \bar{w}, \quad (17)$$

where  $\bar{c}$  and  $\bar{w}$  are to be determined.

A crucial feature is that both non-discretionary income and expenditure as well as the real rate of return are stochastic.  $x_{1,s}$ ,  $x_{2,s}$ , and  $r_s$  all follow autocorrelated log-normally distributed processes:

$$\begin{aligned} \ln(x_{1,s}) &= (1 - \kappa_1) \bar{\ln} x_1 + \kappa_1 \ln(x_{1,s-1}) + u_{1,s+1} \\ \ln(x_{2,s}) &= (1 - \kappa_2) \bar{\ln} x_2 + \kappa_2 \ln(x_{2,s-1}) + u_{2,s+1} \\ \ln(r_s) &= (1 - \kappa_r) \bar{\ln} r + \kappa_r \ln(r_{s-1}) + u_{r,s+1} \\ u_{r,s+1} &= \theta u_{1,s+1} + u_{3,s+1} \end{aligned} \quad (18)$$

Defining

$$\mathbf{u}_s \equiv (u_{1,s}, u_{2,s}, u_{r,s}) \stackrel{\text{d}}{=} N(0, \mathbf{D}), \quad (19)$$

with  $\mathbf{D} \equiv [d_{ij}]$ . The last line of 18 stipulates that real rates vary counter or procyclically with surprises in non-discretionary natural resource revenues, through varying  $\theta = \frac{\text{Cov}_t[r_{t+1}, u_{1,t+1}]}{\text{Var}_t[u_{1,t+1}]}$ .

The risky steady state values are the unconditional means of the ergodic distributions, which up to a second-order approximation are:

$$\begin{aligned} \bar{x}_1 &= e^{\left( \bar{\ln} x_1 + \frac{d_{11}}{2(1-\kappa_1^2)} \right)} \\ \bar{x}_2 &= e^{\left( \bar{\ln} x_2 + \frac{d_{22}}{2(1-\kappa_2^2)} \right)} \\ \bar{r} &= e^{\left( \bar{\ln} r + \frac{d_{33}}{2(1-\kappa_r^2)} \right)} \end{aligned} \quad (20)$$

as in Granger and Newbold (1976). Expressions for the conditional mean and variances of these exogenous processes that are needed to solve the model are presented in appendix A.

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<sup>15</sup>This is akin to normalising by dividing by non-resource GDP.

## 4 Interpretation of the real return on net assets

I have assumed that there is such a thing as the real return on net assets, that part of this return is stochastic and exogenous and that the other part depends on net worth with a negative elasticity. The reader might welcome some more detail and clarification on these points.

Let us assume that the public capital stock  $A_{t-1}$  produces a service flow  $Y_t$  according to a Cobb-Douglas production function:

$$Y_t = r_{a,t}(A_{t-1})^\zeta(\Lambda_t L_t)^{1-\zeta-\mu}(W_{t-1})^\mu. \quad (21)$$

where there are constant returns in physical capital,  $A_{t-1}$ , and labour,  $L_t$ .  $r_{a,t}$  is an exogenous productivity shock (to be fully specified below). An unusual feature is that previous net worth is an input into production as working capital because the greater the net worth, the easier it is to deal with unanticipated expenditures. For example, under duress, while a government might contemplate raising finance through privatisations, that might only translate into retired liabilities at a poor conversion rate (because of firesaling). For simplicity, I assume that this is an externality.<sup>16</sup>

The marginal product of capital using the production function 21 is

$$\begin{aligned} r_{k,t+1} &\equiv r_{a,t+1}\zeta(A_t)^{\zeta-1}(\Lambda_{t+1}L_{t+1})^{1-\zeta-\mu}(W_t)^\mu \\ &= r_{a,t+1}\zeta(\vartheta_{1,t})^{\zeta-1}(\Lambda_{t+1}L_{t+1})^{1-\zeta-\mu}(W_t)^{\mu+\zeta-1} \\ &= r_{a,t+1}\zeta(\vartheta_{1,t})^{\zeta-1}\left(\frac{W_t}{\tau}\right)^{\mu+\zeta-1} \end{aligned} \quad (22)$$

where  $\vartheta_{1,t}$  is the share of time  $t$  net wealth held in physical capital.

To finance production the government invests in net risky financial assets paying a gross interest rate ( $r_{d,t+1}$ ) and an asset with a risk-free gross return ( $r_f$ ), fixed for simplicity. A negative investment in the risky liquid asset is interpreted as debt being greater than the total of assets such as foreign reserves and wealth funds.

$r_{a,t+1}$  and  $r_{d,t+1}$  follow jointly distributed autocorrelated log-normal process :

$$\begin{aligned} lr_{a,t+1} &= \kappa_a lr_{a,t} + u_{a,t+1} \\ lr_{d,t+1} &= (1 - \kappa_d)lr_f + \kappa_d lr_{d,t} + u_{d,t+1} \end{aligned} \quad (23)$$

with  $lr_{x,t+1} \equiv \ln(r_{x,t+1})$  for  $x = (a, d, f)$ .  $u_{a,t+1}$  and  $u_{d,t+1}$  are normally distributed variables with means of zero, respective variances of  $d_{aa}$  and  $d_{dd}$  and a covariance  $d_{ad}$ . The excess log returns to net liquid assets and capital are on average equal to the risk-free rate, by arbitrage.

The budget constraint of the state is as in equation 15 but with net worth now explicitly disaggregated into net financial assets and public capital:

$$\begin{aligned} C_{t+1} &= ((1 - \vartheta_{1,t} - \vartheta_{2,t})r_f + \vartheta_{1,t}(r_{k,t+1} - \delta_k) + \vartheta_{2,t}r_{d,t+1})W_t - W_{t+1} + X_{1,t+1} - X_{2,t+1} \\ &\Rightarrow C_{t+1} = r_{p,t+1}W_t - W_{t+1} + X_{1,t+1} - X_{2,t+1} \end{aligned} \quad (24)$$

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<sup>16</sup>In assessing the social optimal, I continue to take this to be an externality.



where  $\vartheta_{2,t}$  is the share of time  $t$  net wealth represented by net financial assets. This will be negative if debt is greater in amount than financial assets.  $r_{p,t+1}$  is the gross return on the government's portfolio defined as

$$\begin{aligned}\frac{r_{p,t+1}}{r_{f,t+1}} &= 1 + \vartheta_{1,t}\left(\frac{r_{k,t+1} - \delta_k}{r_f} - 1\right) + \vartheta_{2,t}\left(\frac{r_{d,t+1}}{r_f} - 1\right) \\ \frac{r_{p,t+1}}{r_{f,t+1}} &= 1 + \vartheta_{1,t}(e^{lr_{k,t+1} - \delta_k - lr_f} - 1) - \vartheta_{2,t}(e^{lr_{d,t+1} - lr_f} - 1).\end{aligned}\quad (25)$$

Taking logs of the above,

$$lr_{p,t+1} - lr_f = \ln(1 + \vartheta_{1,t}(e^{lr_{k,t+1} - \delta_k - lr_f} - 1) - \vartheta_{2,t}(e^{lr_{d,t+1} - lr_f} - 1)).\quad (26)$$

Define a vector of excess returns as

$$\mathbf{lr}_{s,t+1} \equiv \begin{bmatrix} lr_{k,t+1} - \delta_k - lr_f \\ lr_{d,t+1} - lr_f \end{bmatrix}.\quad (27)$$

Then

$$\begin{aligned}\mathbb{E}_t[\mathbf{lr}_{s,t+1}] &= \begin{bmatrix} \mathbb{E}_t[lr_{k,t+1}] - \delta_k - lr_f \\ \mathbb{E}_t[lr_{d,t+1}] - lr_f \end{bmatrix} \\ &= \begin{bmatrix} \kappa_a lr_{a,t} + \ln(\zeta(\vartheta_{1,t})^{\zeta-1}(\frac{w_t}{\tau})^{\mu+\zeta-1} - \delta_k - lr_f) \\ \kappa_d(lr_{d,t} - lr_f) \end{bmatrix}\end{aligned}\quad (28)$$

and

$$\mathbf{Var}_t[\mathbf{lr}_{s,t+1}] = \begin{bmatrix} \mathit{Var}_t[lr_{k,t+1}] & \mathit{Cov}_t[lr_{k,t+1}, lr_{d,t+1}] \\ \mathit{Cov}_t[lr_{k,t+1}, lr_{d,t+1}] & \mathit{Var}_t[lr_{d,t+1}] \end{bmatrix}\quad (29)$$

$$= \begin{bmatrix} d_{aa} & d_{ad} \\ d_{ad} & d_{dd} \end{bmatrix}.\quad (30)$$

Consider a first-order approximation of  $lr_{p,t+1}$  with respect to  $\mathbf{lr}_{s,t+1}$  about  $\mathbf{0}$

$$lr_{p,t+1} \approx (1 - \vartheta_{1,t} - \vartheta_{2,t})lr_f + \vartheta_{1,t}(lr_{a,t+1} + \ln(\zeta(\vartheta_{1,t})^{\zeta-1}) + (\mu + \zeta - 1)\ln(\frac{w_t}{\tau}) - \delta_k) + \vartheta_{2,t}lr_{d,t+1}\quad (31)$$

such that the gross portfolio return is approximately the product of the exogenous return and an endogenous component, just as in equation 15:<sup>17</sup>

$$\begin{aligned}r_{p,t+1} &\approx r_{r,t+1}\left(\frac{w_t}{\bar{w}}\right)^{-\delta}. \\ \text{where } r_{r,t+1} &= r_f^{1-\vartheta_{1,t}-\vartheta_{2,t}} r_{a,t+1}^{\vartheta_{1,t}} r_{d,t+1}^{\vartheta_{2,t}} e^{-\delta_k \vartheta_{1,t}} \zeta^{\vartheta_{1,t}} (\vartheta_{1,t})^{\vartheta_{1,t}(\zeta-1)}, \\ \delta &= \vartheta_{1,t}(1 - \zeta - \mu) \\ \text{and } \bar{w} &= \tau.\end{aligned}\quad (32)$$

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<sup>17</sup>This interpretation depends on the accuracy of the first-order approximation and portfolio shares being updated infrequently. Campbell, Chan, and Viceira (2003) implement a second-order approximation in a version of this problem and solve for the portfolio shares, but only in the absence of stochastic income.

If debt dominates financial assets ( $\vartheta_{2,t} < 0$ ) and there is a rise in rate of return on risky financial claims (unrelated to the productivity shock  $r_{a,t}$ ) then the exogenous component of the rate of return on net assets ( $r_t$ ) will fall. Under these circumstances, a countercyclical policy that raises rates on risky financial claims when windfall revenues are high connotes a *negative* conditional dependence between the real rate on net assets and revenue. Conversely, procyclicality implies a positive conditional dependence between the real rate on net assets and non-discretionary revenue. This is the interpretation I follow in the rest of the text.

Taking expectations conditional on period t information,

$$Var_t[lr_{r,t+1}] \approx \vartheta_{1,t}^2 d_{aa} + \vartheta_{2,t}^2 d_{dd} + 2\vartheta_{1,t}\vartheta_{2,t}d_{ad}. \quad (33)$$

Equation 33 links limited diversification to risk, just as in standard portfolio theory. Poor diversification in this context is equivalent to a more negative covariance between lending costs and investment returns such that for example a lower rate on investments is more likely to be associated with creditors raising their offered lending rates. We would expect an indebted government that operates in a shallow financial market to face a more unpredictable returns on its net assets. Indeed in equation 33, the more negative  $d_{ad}$ , the larger conditional variance of the log returns on net assets providing that debt dominates liquid financial assets ( $\vartheta_{1,t}$ ).

Turning now to the interpretation of the endogenous component in equation 32, the elasticity of the quantity of net assets on the endogenous component of the return on net assets,  $\delta$ , is a combination of two opposing forces of diminishing marginal returns to capital and the beneficial effect of having higher net worth on production.

But how can this parameter be calibrated? The first influence is equal to one minus the share of capital in nominal public output ( $1-\zeta$ ) multiplied by the share of public capital in net worth  $\vartheta_{1,t}$ . As the share of government spending on GDP is about 40% and the factor share of public capital in total GDP was estimated by Gupta, Kangur, Papageorgiou, and Wane (2011) to be about 20%,  $(1-\zeta)$  should be about  $0.5 = \frac{20}{40}$ . But  $\vartheta_1$  is an optimal share and is therefore difficult to estimate given political impatience. And neither are there, as far as I know, direct estimates of the importance of net worth to production,  $\mu$ , although Lipschitz, Messmacher, and Mourmouras (2006)'s estimates of the beneficial impact of reserves on debt financing costs suggest that it could be substantial. In what follows I take the view that that  $\delta$  is positive, but only just.

## 5 Solution and further interpretation

The first-order Euler equation for maximising the objective 14 subject to the constraint 16 is

$$f(c_{t+1}, r_{t+1}, c_t) \equiv \beta(1-\delta)\mathbb{E}_t\left[\left(\frac{c_{t+1}^\tau}{c_t}\right)^{-\gamma} r_{t+1} \left(\frac{w_t}{\tau \tilde{w}}\right)^{-\delta}\right] - 1 = 0, \quad (34)$$

written in model units. The transversality condition is such that

$$\lim_{s \rightarrow \infty} \beta^s W_{t+s} (C_{t+s})^{-\gamma} = 0 \quad (35)$$

which after expressing in model units and substituting from the Euler equation becomes

$$\lim_{s \rightarrow \infty} \mathbb{E}_t w_{t+s}^{1-\delta} \tau^{(1-\delta)s} C_t r_t \prod_{v=0}^s \frac{w_{t+v}^\delta}{r_{t+v}} = 0, \quad (36)$$

and implies  $(1 - \delta)\bar{r}(\frac{\bar{w}}{\tau\bar{w}})^{-\delta} > \tau$ .

This solvency condition — a lower bound on the real rate adjusted for default risk — is necessary to ensure that the government cannot increase net assets faster than the real returns it pays on them. It is widely recognised that that the solvency condition in certainty equivalent models, simply  $\bar{r} > \tau$ , tells us little about the sustainability of fiscal dynamics (Manasse and Roubini (2009)). With an endogenously determined probability of default and the application of the risky steady state concept, solvency now also depends on the uncertainties faces by the exporting country. As we shall see, an additional impatience restriction is needed to ensure that there exists a positive flow of discretionary expenditure on average, which places an upper bound on the real rate.

A second-order approximation of the first-order condition  $f$  (defined in equation 34) yields

$$\begin{aligned} \mathbb{E}_t [f(c_{t+1}, r_{t+1}, c_t, w_t)] &\approx \hat{\Phi}(\mathbb{E}_t[c_{t+1}, r_{t+1}, c_t, w_t]) \\ \Rightarrow 1 - \underbrace{\left(\frac{\mathbb{E}_t[c_{t+1}]\tau}{c_t}\right)^\gamma \frac{1}{\beta\mathbb{E}_t[r_{t+1}]}}_{\text{Smoothing}} &\underbrace{\frac{1}{1-\delta}\left(\frac{w_t}{\tau\bar{w}}\right)^\delta}_{\text{Capital}} - \underbrace{\gamma \frac{\text{Cov}_t[c_{t+1}, r_{t+1}]}{\mathbb{E}_t[c_{t+1}]\mathbb{E}_t[r_{t+1}]}}_{\text{Risk Premium}} + \underbrace{\frac{\gamma(\gamma+1)}{2} \frac{\text{Var}_t[c_{t+1}]}{\mathbb{E}_t[c_{t+1}]^2}}_{\text{Prudence}} \approx 0, \end{aligned} \quad (37)$$

with the complete derivation in appendix B.

The first term in equation 37 ( $\left(\frac{\mathbb{E}_t[c_{t+1}]\tau}{c_t}\right)^\gamma \frac{1}{\beta\mathbb{E}_t[r_{t+1}]}$ ) is all what we would see in a standard (linearised) description of the fiscal policy problem. As Harding and van der Ploeg (2009) explain, it is optimal to deviate from the Hartwick rule and spend some of the windfall now. How much smoothing takes place depends on the contribution that the resource makes to permanent income. If it is expected to be a temporary boom, there will be little current spending.

The next term reflects the importance of capital in lowering the rate of return. On one hand, more capital lowers the marginal productivity of public capital. On the other, more capital potentially represents more net worth, supporting returns.

The last two terms appear because uncertainty matters.<sup>18</sup> The third term is related to risk aversion while the fourth is related to prudence. Consistently, while the second derivative of the utility function — an elasticity of  $\gamma$  — determines a preference for risk aversion, the third derivative — here an elasticity of one plus the coefficient of risk aversion, or  $\gamma + 1$  — determines the proclivity for prudence (Kimball (1990)).<sup>19</sup> In order to provide some

<sup>18</sup>In consumption based asset pricing models, consumption is considered exogenous and the price of risky assets is determined within an equation similar to 37 with uncertainty terms.

<sup>19</sup>In more general utility functions, it is possible to have these parameters determined independently (van der Ploeg (2010)).

intuition, I consider equation 37 at the risky steady state:

$$\bar{c}^2 \hat{\Phi} = \left(1 - \frac{\tau^\gamma}{\beta(1-\delta)\bar{r}} \left(\frac{\bar{w}}{\tau\tilde{w}}\right)^\delta\right) \bar{c}^2 - \gamma Cov_t [c_{t+1}, r_{t+1}] \frac{\bar{c}}{\bar{r}} + \frac{\gamma(\gamma+1)}{2} Var_t [c_{t+1}] \approx 0. \quad (38)$$

This quadratic formula in  $\bar{c}$  will have real solutions if

$$\beta(1-\delta)\bar{r} \left(\frac{\bar{w}}{\tau\tilde{w}}\right)^{-\delta} < \tau^\gamma. \quad (39)$$

39 is an impatience condition, an intrinsic feature of the buffer stock problem (Carroll (2004)). If the rate of return on assets, the growth rate is too low, the government is too risk averse or does not discount the future highly enough, then there will be no risky steady state because the government will only keep saving.<sup>20</sup>

Combining with the solvency condition (36) gives a range

$$\beta(1-\delta)\bar{r}\tau^{-\gamma} < \left(\frac{\bar{w}}{\tau\tilde{w}}\right)^\delta < \frac{(1-\delta)\bar{r}}{\tau}. \quad (40)$$

If we take the view that the wider the gap between these bounds, the greater the comfort zone for an optimal solution to exist, then  $\frac{(1-\delta)\bar{r}}{\tau}(1-\beta\tau^{1-\gamma})$  provides a measure of sustainability. The comfort zone is wider, the larger the (ergodic) mean rate of return on net assets ( $\uparrow \bar{r}$ ), the more impatience in policy ( $\uparrow \beta$ ), the less diminishing marginal returns compared to financial frictions ( $\downarrow \|\delta\|$ ), and assuming there is prudence ( $\gamma > 1$ ), the less of it ( $\downarrow \gamma - 1$ ) and the higher the growth rate per capita ( $\uparrow \tau$ ). In what follows, these bounds are always satisfied.

Following section 2, the next step is to posit that the only state variable,  $w_t$ , follows the expectations rule

$$w_{t+1} = \bar{w} + G_{ww}(w_t - \bar{w}) + G_{wr}(r_{t+1} - \bar{r}) + G_{w1}(x_{1,t+1} - \bar{x}_1) + G_{w2}(x_{2,t+1} - \bar{x}_2), \quad (41)$$

where the coefficients  $G_{ww}$ ,  $G_{wr}$ ,  $G_{w1}$  and  $G_{w2}$  as well the risky steady state value of net assets  $\bar{w}$  are to be determined. When combined with the budget constraint, this gives

$$\begin{aligned} c_{t+1} = & (1 - G_{w1})x_{1,t+1} - (1 + G_{w2})x_{2,t+1} \\ & + \left(\frac{w_t}{\tau} \left(\frac{w_t}{\tau\tilde{w}}\right)^{-\delta} - G_{wr}\right)r_{t+1} - G_{ww}w_t \\ & + (G_{ww} - 1)\bar{w} + G_{wr}\bar{r} + G_{w1}\bar{x}_1 + G_{w2}\bar{x}_2. \end{aligned} \quad (42)$$

Equation 42 clarifies that the expression 37 is a partial description of our solution. Not only is the term in expected consumption is endogenous as in a linearised Euler equation but also the variance and covariance need to be solved out jointly with the risky steady

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<sup>20</sup>This is only for the risky steady state. There is another condition in dynamics: see Arrau and Claessens (1992) for an example. The impatience restriction and the transversality condition 36 can be mutually satisfied providing we have sufficient growth, not too much risk aversion and not too high a social rate of discount ( $(1-\gamma)\tau > \beta$ ).

state and the rational expectation coefficients. Nevertheless we can use economic intuition to guide the values of these coefficients. First we could expect that  $0 < G_{ww} < 1$ , as then net assets are more likely to converge. Second and third,  $0 < G_{w1} < 1$ ,  $-1 < G_{w2} < 0$ , but with both being closer to one in absolute value, because this implies that most of extra oil revenue will be saved on accrual and that extra expenses are not offset one for one with lower discretionary expenditure. In fact, we shall see that the values of these coefficients do respect these bounds, even for a wide range of parameter values where the solution converges.

If  $\frac{w_t}{\tau} \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} > G_{wr} > 0$ , then we would expect that governments which are net savers to respond to lower returns on net assets by decreasing discretionary expenditure (adding to net asset stocks). With public capital included in  $w$ , most governments will be net savers. This would then potentially be an important transmission channel by which through raising debt interest rates and lowering the return on net assets, a countercyclical rate policy stimulates a build up of net assets in boom times. But with  $\delta$  even slightly positive and high level of net assets, governments may be bound by diminishing returns to capital to do the opposite, save less when rates of return fall. The model therefore contains the inherent constraint on a countercyclical interest rate policy represented by diminishing marginal returns.

In the following sections I quantify the strength of these effects. But before that I establish a metric for quantifying the importance of uncertainty.

## 6 A welfare metric

In this section, I explain how I judge outcomes in terms of both the mean and conditional variance of consumption, through an approximation for utility-based social welfare function. As I shall show, it makes a difference that steady state levels in this framework are responsive to uncertainty parameters as they come to play a dominating role in the welfare comparison. Crucially as this is for welfare, the discount rate is  $\hat{\beta}$  and can be lower than that used by the government in formulating its fiscal plans.

A second-order approximation of the utility function in equation 14 is as follows:

$$\begin{aligned}
\mathbb{E}_t U(C_t, \dots, C_\infty) &\approx V(\mathbb{E}_t c_t, \dots, \mathbb{E}_t c_\infty) \\
&\equiv L_t \sum_{s=t}^{\infty} \hat{\beta}^{s-t} \frac{L_s}{L_t} \tau^{(s-t)(1-\gamma)} \frac{\bar{c}^{1-\gamma} - 1}{(1-\gamma)} \\
&+ L_t \sum_{s=t}^{\infty} \hat{\beta}^{s-t} \frac{L_s}{L_t} \tau^{(s-t)(1-\gamma)} \bar{c}^{-\gamma} (\mathbb{E}_t [c_s] - \bar{c}) \\
&- \frac{1}{2\gamma} L_t \sum_{s=t}^{\infty} \hat{\beta}^{s-t} \frac{L_s}{L_t} \tau^{(s-t)(1-\gamma)} \bar{c}^{-(1+\gamma)} \text{Var}_t [c_s] \\
&= \frac{(\bar{c})^{1-\gamma} - 1}{(1-\gamma)(1 - (1+n)\tau^{1-\gamma})} \\
&- \frac{\gamma \bar{c}^{-(1+\gamma)}}{2} \sum_{s=t}^{\infty} \hat{\beta}^{s-t} (1+n)^{s-t} \tau^{(s-t)(1-\gamma)} \text{Var}_t [c_s]. \tag{43}
\end{aligned}$$

I am assuming that the population grows at a constant rate and is normalised at  $L_t = 1$ . Note also that the term in the expected deviation of discretionary expenditure from its risky steady state mean is dropped as discretionary expenditure are assumed to begin at the risky steady state, and is expected to be there on average forever.

Being a convolution of jointly log-normal variables, the process  $c_t$  is  $h$ -step ( $h > 1$ ) conditionally heteroscedastic and it is not easy to determine the conditional variance of future consumption,  $\text{Var}_t [c_{t+s}]$ . I propose to approximate welfare further by

$$\begin{aligned}
\mathbb{E}_t U(C_t, \dots, C_\infty) &\approx V_2(\mathbb{E}_t c_t, \dots, \mathbb{E}_t c_\infty) \\
&\equiv \frac{(\bar{c})^{1-\gamma} - 1}{(1-\gamma)(1 - \hat{\beta}(1+n)\tau^{1-\gamma})} \\
&- \frac{\gamma \bar{c}^{-(1+\gamma)}}{2} \sum_{s=t}^{\infty} \hat{\beta}^{s-t} (1+n)^{s-t} \tau^{(s-t)(1-\gamma)} \text{Var}_t [\hat{c}_s] \tag{44}
\end{aligned}$$

where  $\text{Var}_t [\hat{c}_s]$  is the variance of an approximation to consumption calculated using linearised versions of the budget constraint and the exogenous processes. This approximation is derived in appendix C and exactly equal to the analytical expression in the next period (when  $s = t + 1$ ).

Expression 44 balances the gains of a high mean level of discretionary expenditure against the benefits of lower volatility within the metric of utility-based welfare. At first glance, this seems similar to the standard approximation around a deterministic steady state (Fernandez-Corugedo (2004)). However as the scaling factor on the volatility term ( $-\frac{\gamma \bar{c}^{-(1+\gamma)}}{2}$  in 44) depends on the mean value of discretionary expenditure and through that, uncertainty parameters, in the risky steady state framework, welfare evaluation is more of a comparison between different steady states rather than a comparison around a fixed state.<sup>21</sup>

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<sup>21</sup>This means that we cannot isolate the welfare cost of volatility. Following Lucas (2003), the welfare

I assess total welfare across different scenarios by an *ex-ante compensating variation* defined as the value of an increase in discretionary expenditure (fixed in percentage terms) that is expected to yield the same utility as in the base-line scenario. This is the value of  $\alpha$  that solves the following equation:

$$(1 + \alpha)^{1-\gamma} V_2(\mathbb{E}_t c_t, \dots \mathbb{E}_t c_\infty) = V_2(\mathbb{E}_t \hat{c}_t, \dots \mathbb{E}_t \hat{c}_\infty). \quad (45)$$

where  $\hat{c}_s$  defines the baseline expenditure stream. One criticism to this could be that the approximation errors involved under each scenario may differ in magnitude and sign to that under the baseline, and thus the welfare comparison could be severely distorted. Given the size of the welfare differences I find, the approximation errors would individually have to be very large for that to be true. Nevertheless this can be investigated in future work.

## 7 Calibrations

The parameter values are chosen to match a risky steady state for Colombia at an annual frequency. Indeed Colombia presents a good case study for this approach. For while it is not among the countries completely dominated by its energy sector, energy and coal production matters for macroeconomic outcomes. External sales from petroleum and other mineral products have risen to 69% by the end of 2011 from 40% of exports in 2001. The dominance of energy is also reflected in the capital account: 73% of the FDI inflow into Colombia in 2010 was destined for mineral and energy sector. More saliently, commodity-related revenues also play a large role in fiscal policy. Between 2007 and 2010, central government received 1.4%, and regional governments, 1.2%, of GDP per year on average from energy and mineral royalties. As the flow of oil extracted from Colombian soil is set to keep increasing at least until 2020 and, if the oil price remains not too far below its current price of nearly 100 dollars a barrel, these fiscal revenues should become even more important — reaching 3 to 4% of GDP until 2020. From then on, one possible scenario is they will peter out rapidly. Other forecasts are for continued strong revenue streams. Yet another scenario is that a deteriorating security situation renders the costs of extracting and piping oil prohibitive curtailing the windfall. The challenge is to make the most of this possibly temporary but certainly uncertain windfall and significantly improve the wellbeing of current and future generations of Colombians.

As the first calibration, I used Iregui and Melo (2010)'s estimate of  $\gamma$ , the coefficient of risk aversion for Colombia, which is about 2.5 and so indicates positive prudence.<sup>22</sup> My

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cost of volatility would be defined as the value of a time-invariant, percentage increase in (growth-adjusted) risky discretionary expenditure that would yield the same utility as given by mean discretionary expenditure without any uncertainty. But there is no defined equilibrium value of net assets without volatility. And with volatility, the mean level of expenditure varies across ergodic steady states. While we can compare the welfare outcomes of different scenarios each with different ergodic steady states, we cannot calculate a volatility contribution to the difference in welfare.

<sup>22</sup>Their estimate has the merit of conditioning on the limited access of the many poorer Colombians to financial services. It also falls within the range of the previous estimates they survey.

calibration of the long-run growth rate for Colombia will unavoidably be controversial. In the 1990s, Colombian GDP grew at a low average annual rate of 2.9%. But this recovered to 4.1% over the proceeding decade and not all of this extra increase is even mechanically due to energy, whose contribution to GDP increased by only a percentage point between the two decades.<sup>23</sup> Hence I choose 4% — a rate close to a fifty year historical average — as an estimate of the long-run growth rate in the absence of temporary resource windfalls.

The mean real interest rate on net government financial assets was set at 10% (in log terms). In keeping with equation 33 in section 4, this reflects a weighted average of the real rate on public capital and the rate of return of net financial assets. The rate of return on net financial assets is broadly consistent with Colombia’s recent external debt spread over US treasury bills, earnings of foreign assets, both adjusted for appreciation, the domestic currency bond rate as measured by Galindo and Hofstetter (2008) for 2002-2006 and an estimated rate of earnings on government’s bank deposits. The rate of return on public capital was based on the mean rate of 12% for World Bank projects estimated by Pohl and Mihaljek (1992) adjusted for a 2.5% depreciation rate used by Gupta, Kangur, Papageorgiou, and Wane (2011).

Galindo and Hofstetter (2008) estimate the unconditional variance of domestic bond rates at 7pp for 2002-2006. And I calculated the unconditional variance of the real external finance rate using World Bank data to be of the same magnitude. These are surprising volatile, quite possibly more volatile than we could expect for the future of Colombian state debt. On the other hand my aim is to estimate the volatility of the rate of return in net assets, allowing for the rate of return on public capital. Thus a conservative assumption would be that the unconditional variance of returns to net assets including public capital is 7pp. The autoregressive parameter on a simple regression of real lending rates was 0.9 (much of Colombian debt is of long maturity), which if we can carry over to assets, implies a conditional standard deviation of real returns of 3pp =  $(7^2 \times (1 - 0.9^2))^{0.5}$ . This implies very unpredictable returns on net assets: the 95% confidence interval of next year’s rates of return is 4 to 17% (in gross rate terms).

The means, conditional variances and persistence of non-discretionary fiscal revenues and expenditures were calibrated using fiscal data as follows. First I took annual data series on the nominal peso revenues of the consolidated Colombian public sector from the Banco de la Republica from 1986 to 2010, and divided them by annual nominal GDP. The average of this series was taken to be the mean level,  $\bar{x}_1$ . I regressed the log of this series on a constant, a lag of itself, the lagged GDP output gap and the detrended world oil price cycle. The standard deviation of this regression became a measure of the conditional volatility,  $\sqrt{\bar{d}_{11}}$ , while the coefficient on the lagged dependent variable was used for  $\kappa_1$ . This implies a 95% confidence interval for next period revenues of between 24.5 and 30.7% of GDP.

I then regressed the log of the ratio of nominal non-interest expenses of the consolidated Colombian public sector to GDP on a constant, its lagged value, the lagged GDP output gap and the log of revenues to GDP. The average of the dependent variable was taken to be

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<sup>23</sup>I am using World Bank data on the contribution of resource rents to GDP in nominal terms to make this rough calculation. Growth in the 1990s was affected by a boom and bust cycle.



$\bar{x}_2$  and the standard deviation of the regression was taken to be  $\sqrt{d_{22}}$ . The coefficient on the lagged dependent variable was used for  $\kappa_2$ . Covariances were assumed to be zero in the baseline case.<sup>24</sup> Table 4 in appendix C summarises these values.

$\delta$  was set at 0.01, following the arguments Section 4 and  $\tilde{w}$  was chosen to be 1, with the justification of these calibrations deferred to later on. I set the social impatience  $\hat{\beta}$  to discount at 2% and population growth to be 1.5%. At these values, both the transversality condition and the impatience restriction are satisfied.

These should be taken to be rough calibrations. Nevertheless, it is reassuring that the solution values are reasonably robust to different values within a plausible range of most of the parameters. The exceptions were  $\delta$  and  $\tilde{w}$ , whose values we have little prior idea and to which the solution can be sensitive. Shortly I discuss how I justify the calibrations for these parameter values.

In appendix B I explain how I apply the method outlined in 2 to obtain numerical solutions to  $G_{ww}$ ,  $G_{wr}$ ,  $G_{w1}$ ,  $G_{w2}$ ,  $\bar{w}$  and  $\bar{c}$ . The endogenous risky steady state values in our baseline case are contained in Table 1 below.

**Table 1: Outcome of calibrations (base-line)**

Risky steady state values:	N-DR	N-DE	RR
	$\bar{x}_1 = 27.6\%$	$\bar{x}_2 = 25.4\%$	$\bar{r} = 11\%$
Covariances:	N-DR	N-DE	RR
N-DR	$\sigma_{11}^2 = 2.51$	-	-
N-DE	$\sigma_{12} = 0$	$\sigma_{22}^2 = 1.74$	-
RR	$\sigma_{13} = 0$	$\sigma_{23} = 0$	$\sigma_{33}^2 = (3.4pp)^2$
Mean outcomes	Assets	DE	Primary deficit
	$\bar{w} = 28.8\%$ GDP	$\bar{c} = 3.1\%$	$\bar{c} - \bar{x}_1 + \bar{x}_2 = 0.9\%$

The value of optimum net assets is found to be 29% of GDP. To compare this against data, I take public sector capital to be the only non-financial asset of the Colombian government. Gupta, Kangur, Papageorgiou, and Wane (2011)'s estimates are that the ratio between Colombia's real public sector capital stock and real GDP was 0.71 in 2007 at 2005 prices and has been falling. Botero and Ramirez Hassan (2010)'s estimates indicate that the real user cost of capital of all investment has also been falling, at about 6% year in Colombia.<sup>25</sup> Extrapolating these trends gives us an estimate of the nominal share of the public capital stock to GDP of approximately 40% in 2011. The total net financial debt of the consolidated Colombian public sector was 28% in 2011, which would leave net assets at just over 10% GDP. Hence, the estimate in the data is well below our calibrated value of 29%. But the recently announced plans of the Colombian government are indeed to lower net financial debt permanently and to raise the value of the public capital stock. I thus consider 28% to be the optimal level of average net assets, implying a small primary surplus in expectation.

Crucially, it is difficult to get a much *lower* mean level of net assets with any plausible assumptions about mean returns. Thus even if the mean rate of return on net assets were 7%

<sup>24</sup>In fact, I found a strong positive correlation between my crude measures of non-discretionary expenditures and revenues (0.6).

<sup>25</sup>Much (but not all) is due to a one-off fall in the real interest rate that is unlikely to continue.

in terms of log returns (or 7.5% in gross terms), optimal net assets would only fall to around 25%. Raising mean non-discretionary revenues by 20% would indeed lower optimal mean net assets (as a smaller asset cushion is needed) to 10%. But this would be extremely unlikely, at a probability of 0.01% according to a log-normal distribution. The conditional one step ahead standard deviation of net assets in Colombia in the risky steady states is estimated to be 1.9 pp of GDP. For those countries where volatile resource revenue contributes more to fiscal revenues, there would appropriately be a higher standard deviation around the mean value of net assets.

The point is then that the calibration of optimal net assets is quite insensitive below to the choices of mean returns within a plausible range. In the following sections, I experiment with different degrees of uncertainty and also political impatience to isolate the reason why net assets in the Colombia data are much lower than the optimal value.

This asymmetric sensitivity of net assets also applies to the two parameters which are most difficult to calibrate. A smaller value of  $\tilde{w}$  (at 0.01 as opposed to the baseline value of 1) lowers net assets to 25% (while a value of 150 raises mean net assets to 80%). Similarly if  $\delta$  is at  $-0.04$  compared to  $-0.01$  in the baseline, net assets fall to 20%. If  $\delta$  is  $-0.001$ , average net assets rise to 50%. Values of  $\delta$  outside this bound generate unstable solutions depending on the value of  $\tilde{w}$ , as the solvency and impatience conditions (bounds 40) are not satisfied. In the absence of further work to develop out economic understanding of this sensitivity, this supports the baseline choices for these two tricky parameters.

The implied linear conditional dependences between variables are presented in Table 2.

**Table 2: Predicted linear conditional dependences (baseline)**

Of $\rightarrow$	Net assets	Primary deficit	(net) Discretionary Expenditure	Non-discretionary Revenue	Real rate ( $\div 100$ )
	$w_t$	$c_t$	$c_t - x_{1,t} + x_{2,t}$	$x_{1,t}$	$r_t$
On $\downarrow$					
Net assets	-	-0.98	4.90	0.84	0.25
Primary deficit	-0.79	-	-4.49	-0.84	0.02
DE	0.18	-0.21	-	0.16	0.02

Our baseline calibration implies that surprises to net assets will be positively related to non-discretionary (natural resource) revenue surprises, with an multiplier of 0.84: a 1pp surprise rise in revenues should lead to a rise in net assets of the order of 0.84pp of GDP on average. This saving is the combined effect of consumption smoothing, capital and precautionary saving effects. Another indication is that the primary deficit is negatively related to revenue surprises by the same magnitude. This implies that a rise in revenues of 1pp of GDP should imply a 0.16pp increase in discretionary expenditure, and thus not all of the extra revenue is saved when they are realised.<sup>26</sup> Note also that a surprise rise in real return on net assets of 1pp leads to a 0.25pp GDP rise in net assets and a 0.02pp increase

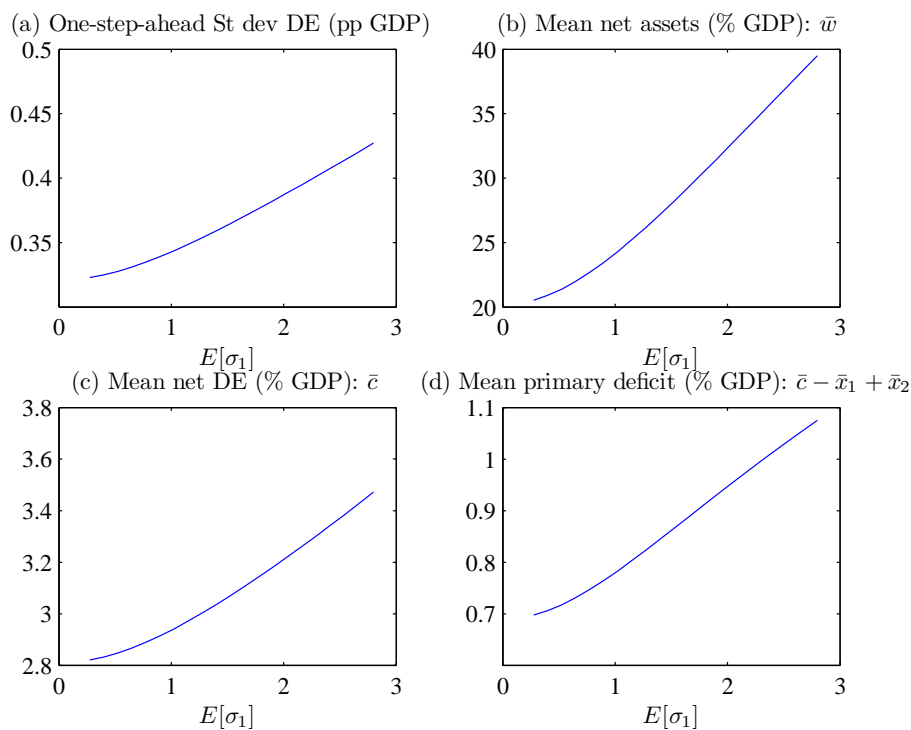
<sup>26</sup>Rincón Castro (2010)'s estimates of the optimal fiscal rule for Colombia also imply substantial counter-cyclicality at an overall level — he favours that a 1pp rise in the total output gap should lead to a concurrent fall in the deficit of 0.3pp of GDP.

of the primary deficit; nearly all of capital market shocks are absorbed into savings.

## 8 Sensitivity to revenue uncertainty

Chart 1 below explores the effect of uncertainty over oil revenues on mean values, an effect that is completely ignored in certainty equivalent frameworks. Revenue volatility ( $\sigma_1$ ) was 1.6pp of GDP in the baseline calibration. The highest value contemplated in Chart 1 is nearly twice that. Panel (d) shows how the one-step ahead conditional variance of discretionary expenditure increases. This is offset by holding more net assets: panel (c) shows how the equilibrium stock of net assets rises. If state revenue is 1pp of GDP more volatile than in the baseline, the optimal mean level of net assets would reach 35% of GDP. Not even with a very predictable revenue stream, can we reach net asset levels of 10%. More net assets imply a higher stream of revenue to sustain discretionary expenditure, and a larger deficit: panels (a) and (b).

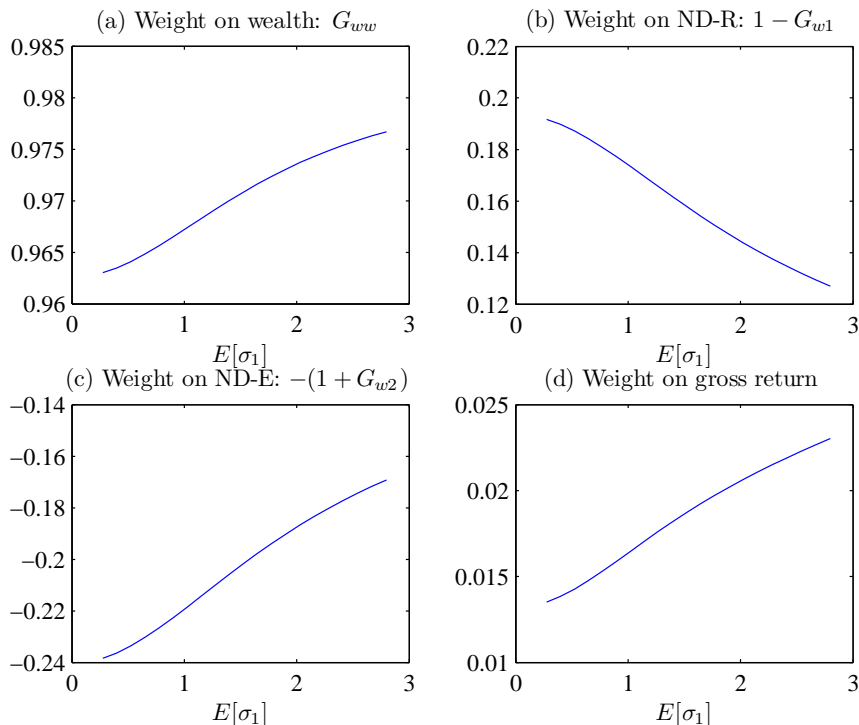
**Chart 1: Effect on risky steady state of revenue uncertainty<sup>a</sup>**



<sup>a</sup> $E[\sigma_1]$  is the average one-step ahead conditional standard deviation of non-discretionary revenues in pps of GDP.

Chart 2 describes the effect on the parameters in the savings rule.

**Chart 2: Effect on parameters in savings rule of revenue uncertainty<sup>a</sup>**



<sup>a</sup> $E[\sigma_1]$  is the average one-step ahead conditional standard deviation of non-discretionary revenues in pps of GDP. The weight on the gross return in panel (d) is  $(\frac{\bar{w}}{\tau}(\frac{\bar{w}}{\tau\bar{w}})^{-\delta} - G_{wr})$ .

The values fall within the ranges suggested in Section 3. The persistence of debt,  $G_{ww}$ , is positive and less than one, but rises with higher volatility reflecting more attenuation as revenue news becomes noisier. The value of  $G_{w1}$  at our baseline standard deviation of oil revenues is consistent with a rule of 0.16pp of an exceptional oil revenue of size 1pp of annual GDP being spent immediately and the rest saved.  $G_{w2}$  in the baseline is such that a sudden 1pp expenditure commitment is only offset with an 0.21pp reduction in discretionary expenditure. As revenue volatility increases, a lower proportion should be spent immediately and likewise, a lower proportion of a surprise expenditure commitment should be absorbed by the discretionary budget. In the baseline, surprises in the real return on net assets are mostly saved: a 1pp rise in rates above the mean value leads to only a 0.02pp of GDP rise in discretionary spending. With greater revenue volatility, there is a larger stock of net assets and a bit more of the rate rise is saved.

The persistence of revenue streams also matters. In the baseline, the autoregressive parameter on the log of non-discretionary revenue is 0.7. If expected streams of non-discretionary revenue are more short-lived — a sequence of bonanzas that come and go

— then volatility falls and a somewhat smaller net asset position is warranted (21% when there is no autoregression). But more is saved from each surprise bonanza, reflecting that each rise is less likely to affect permanent income. Indeed with no autoregression, only 6pp of a 1pp rise in revenue is spent as opposed to 16pp in the baseline. Conversely more persistent revenue streams imply that the country should be less of a saver.

Persistence matters in the spending multiplier. van der Ploeg (2010) provides the following back of the envelope formula for this elasticity:  $\frac{\gamma\sigma_1}{2\bar{x}_1}$ . For the baseline parameters here would imply a value for  $1 - G_{w1}$  of 0.7pp, half the size of the model-based estimate. Yet, the smaller value is though to be expected as the simple formula assumes that shocks are not persistent.<sup>27</sup>

## 9 Sensitivity to asset return uncertainty

In the baseline, the standard deviation of the return on net assets was set at 3pp. In the introduction and in Section 4 I argued that a government that operates in a market with limited financial depth is likely to face more volatile rates. To explore the implications of financial shallowness, I vary the uncertainty in real rates. Higher return uncertainty makes it more costly to use net assets as a cushion, and the size of the net stock held shrinks. However it never falls below 28% of GDP.

Conversely, with very predictable asset returns, the optimal level of net assets rises dramatically. When volatility is at 1pp, the optimal level of net assets is just over 40% of GDP. The multiplier of rates of return onto discretionary expenditure also rises when substantially when return volatility falls, a rise equivalent to the 1pp standard deviation would lead to a 0.08pp rise in discretionary expenditure, four times that of the baseline. This is yet another permanent income effect; only returns that are not likely to be reversed are spent. While these results are illustrative, given that public capital is an important component of net assets, I would consider a 1pp asset return standard deviation to be implausibly low.

## 10 The effect of political impatience

Up until this point, I have assumed that the future is discounted at 2% by policymakers and that this is the social discount rate. In the introduction, I cited many studies which argue that policymakers receiving resources revenue discount the future at a suboptimally high rate.

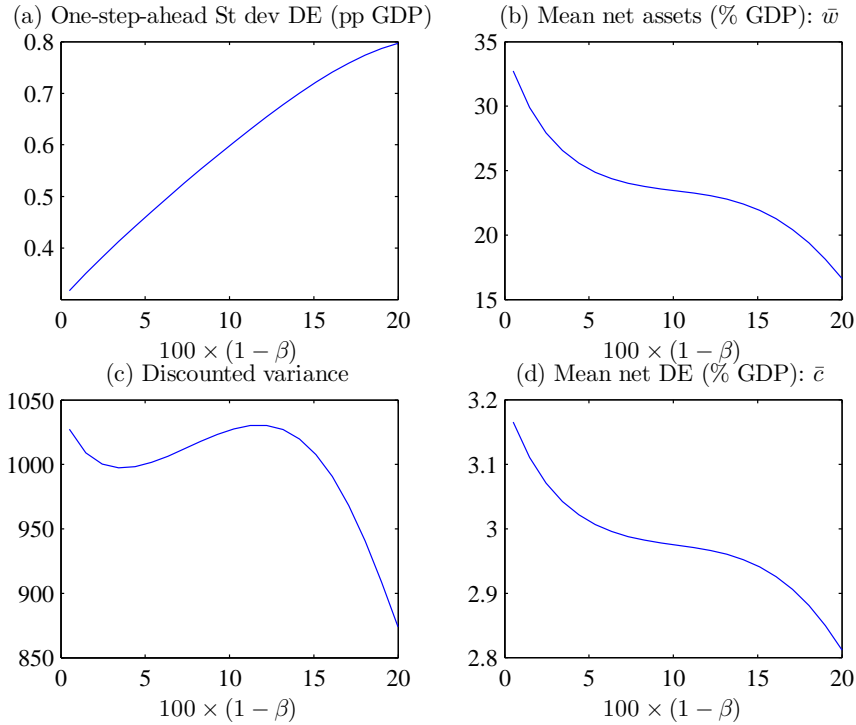
In Chart 3, I describe the effect of more policy short-termism. In panel (d), we see that the one-step ahead conditional variance of discretionary expenditure increases, doubling in scale as the discount rate rises from 2% to 20%. More impatient policymakers hold much lower net assets: the mean value of net assets falls to 16% (panel c). As it was difficult

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<sup>27</sup>Estimates of this elasticity should also crucially depend on the assumption that net assets (which included public investment) generate a positive risk-adjusted return, and not an exorbitant carrying cost.

to obtain such low levels of net assets by adjusting revenue or return volatility, this would suggest that, prima facie, policy impatience is the most likely cause of low buffers.

**Chart 3: Effect on risky steady state of political impatience<sup>a</sup>**

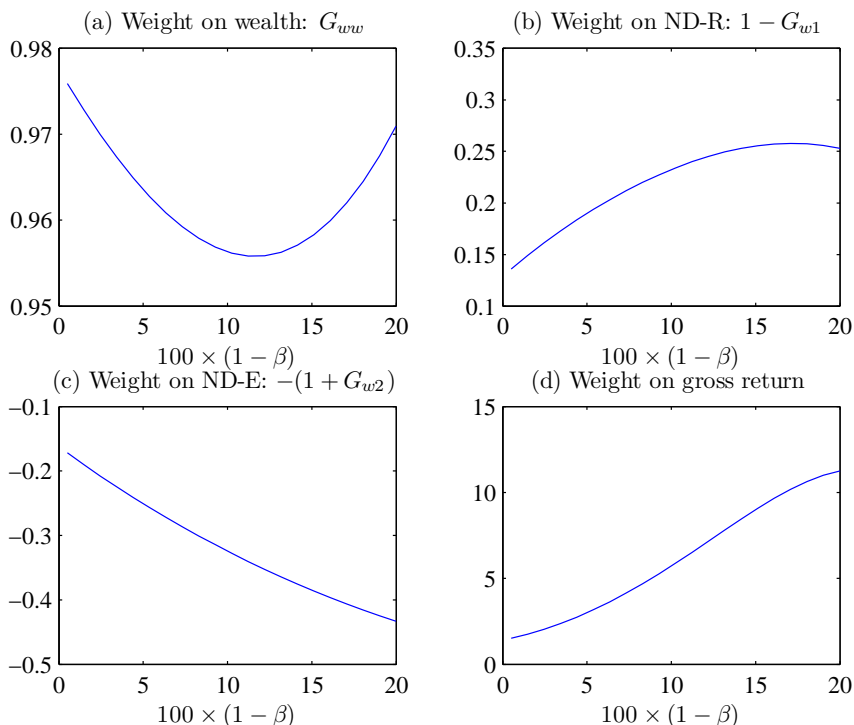



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<sup>a</sup>x-axis describes the discount rate used in formulating policy. Discounted variance in welfare is defined as  $\sum_{s=t}^{\infty} \hat{\beta}^{s-t} (1+n)^{s-t} \tau^{(s-t)(1-\gamma)} Var_t[\hat{c}_s]$  in equation 44.

It is interesting to see how the parameters in the savings rule are also affected: Chart 4.

**Chart 4: Effect on parameters in savings rule of political impatience<sup>a</sup>**



<sup>a</sup>x-axis describes the policy discount rate. The weight on the gross return in panel (d) is  $(\frac{\bar{w}}{\tau} (\frac{\bar{w}}{\tau \bar{w}})^{-\delta} - G_{wr})$ .

With more impatience, there is a significantly greater propensity to adjust spending on accrual. The multiplier of non-discretionary revenues on net discretionary spending rises from 0.16 to 0.26 in the extreme case of a 20% discount rate. Similarly, an immediate liability is met by simultaneously cutting discretionary expenditures by 0.44pp. Moreover, more is immediately spent from unexpected rises in the real return on assets: a 1pp rise in the return leads to 0.12pp rise in discretionary expenditure in the extreme case.

## 11 The effect of pro- and counter-cyclical rates of return

I now consider whether countercyclical borrowing rates can improve outcomes even under the face of substantial natural resource revenue uncertainty. The essence of the countercyclical interest rate policy is that real returns stimulate the use of net assets as a cushion — encouraging saving in good times and spending in bad times. The most natural way that this could be operationalised is to make real return on net assets negatively conditionally linearly

dependent on surprises to resource revenues as in equation 18. In the baseline, real rates of return on the net reproducible assets of the states do not covary with non-discretionary revenues. A more countercyclical policy would make return on net assets covary negatively with revenues while a procyclical bias would imply a positive covariance. Activism, whatever the orientation, raises the unpredictability of real rates.

However the effect of countercyclicity is analytically complex. Consider the following expressions for the conditional next period variances and covariance of discretionary expenditure<sup>28</sup>

$$Var_t [c_{t+1}] = \left(\frac{w_t}{\tau}\right)\left(\frac{w_t}{\tau\tilde{w}}\right)^{-\delta} - G_{wr})^2 \sigma_{r,t}^2 + 2(1 - G_{w1})\left(\left(\frac{w_t}{\tau}\right)\left(\frac{w_t}{\tau\tilde{w}}\right)^{-\delta} - G_{wr}\right)\sigma_{1r,t} \quad (46)$$

and

$$Cov_t [c_{t+1}, r_{t+1}] = \left(\frac{w_t}{\tau}\right)\left(\frac{w_t}{\tau\tilde{w}}\right)^{-\delta} - G_{wr})\sigma_{r,t}^2 + (1 - G_{w1})\sigma_{1r,t}. \quad (47)$$

As explained in section 5, the risky steady state level of discretionary expenditure is determined by the variance of discretionary expenditure and its covariances with rates of return. Expressions 46 and 47 show that while these are in turn affected by the procyclicality through the term  $\sigma_{1r,t}$ , the size and direction of the effect is sensitive to parameter constellations, in particular whether higher returns lead to a lower discretionary expenditure (whether  $\left(\frac{w_t}{\tau}\right)\left(\frac{w_t}{\tau\tilde{w}}\right)^{-\delta} > G_{wr}$ ). Then more activism in rates would raise the volatility of returns relative to the benchmark. Finally for welfare it is not the just the predictability of next period's discretionary expenditures, but all future expenditures that matter. In the face of such theoretical ambivalence, numerical calculations are needed to establish the viability of the policy.

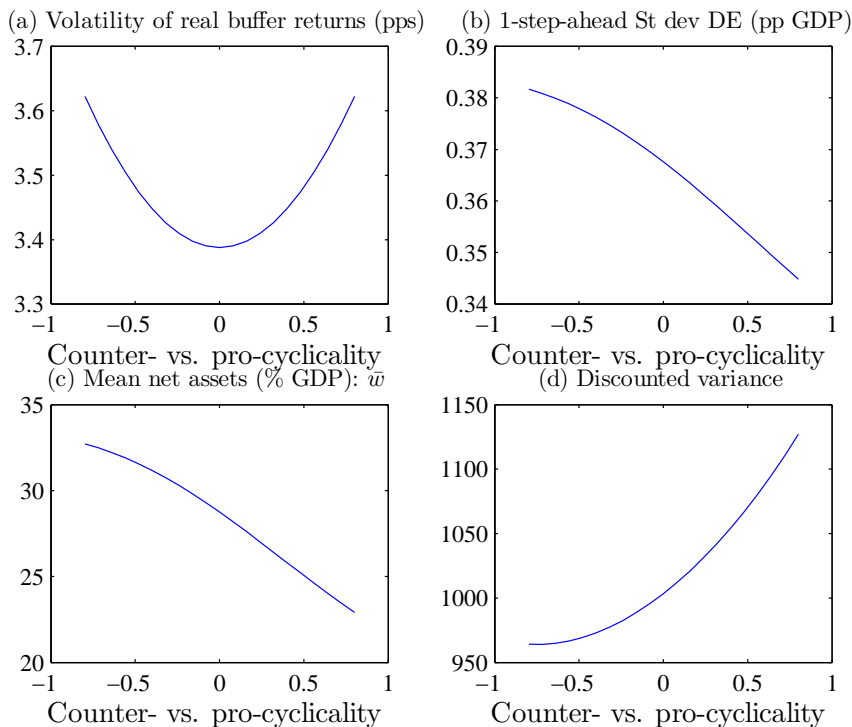
Chart 5 plots the effect of swinging between pro- and counter-cyclicity. At one extreme, there is a countercyclical policy which lowers real returns by 0.8pp when non-discretionary revenue is shocked by 1pp of GDP. At the other, a strong procyclical bias raises real returns by the same magnitude. The baseline lies at the midpoint. As I argue below, we are likely to observe a narrower range of policies in practice.

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<sup>28</sup>The derivation is in appendix B. I have dropped all irrelevant terms when moving 46 and 47 from the appendix.



**Chart 5: Effect on risky steady state of counter- or procyclical returns<sup>1</sup>**



<sup>a</sup>The volatility of asset returns in panel (a) is  $E[\sigma_r]$ . Counter- vs. pro-cyclical means the average linear dependence of real returns on net assets on non-discretionary revenues or  $100 \times \frac{E[\sigma_{1r}]}{E[\sigma_1^2]}$ . Discounted variance in welfare is defined as  $\sum_{s=t}^{\infty} \hat{\beta}^{s-t} (1 + n)^{s-t} \tau^{(s-t)(1-\gamma)} Var_t[\hat{c}_s]$  in equation 44.

Both pro- or countercyclical policies raise the volatility in asset returns relative to the benchmark (see panel (a)) simply through more activism. This effect may seem slight, but as we shall see even a 2 pp rise in return volatility can have important welfare consequences. The countercyclical policy differentially raises the short-run unpredictability of discretionary expenditure (panel (b)), but this effect is relatively small. The most important differential effect of moving on the scale from pro- to counter-cyclical is on the government's balance sheet: net assets rise to 33% of GDP (panel (c)) with extreme countercyclical. This is because a countercyclical policy improves the functioning of the portfolio by making the covariance of revenue and returns more negative (47) and thus lowering the risk premium. The greater level of net assets lowers the variance of discretionary expenditure discounted optimally over the future (panel (d)) offsetting the rise in short-run unpredictability.

## 12 Welfare effects

In this section I compare the different causes against a common metric. Table 3 presents the effects of deviations from the baseline case along the dimensions of political impatience, asset return volatility, counter- or pro-cyclicality and revenue uncertainty, separately.

**Table 3: Comparing explanations**

	Welfare <sup>a</sup>	Net Assets <sup>b</sup>	Weight N-DE <sup>c</sup>	Weight Returns <sup>d</sup>	Smoothness <sup>e</sup>
Political Impatience					
discount rate (%)					
2	0.0	29	0.16	0.02	0.23
20	25.8	17	0.25	0.11	0.50
Buffer Return Volatility					
pp					
3.4	0.0	29	0.16	0.02	0.23
4.0	24.3	28	0.16	0.02	0.23
Pro vs Counter-cyclicality					
$100 \times \frac{\sigma_{1r,t}}{\sigma_{1,t}^2}$					
0.0	0.0	29	0.16	0.02	0.23
0.6	23.9	25	0.15	0.00	0.22
Revenue Uncertainty					
pp GDP					
1.6	0.0	29	0.16	0.02	0.23
3.0	11.1	41	0.12	0.02	0.15

<sup>a</sup>% increase in consumption in perpetuity needed to make welfare equivalent to baseline case. See equation 45.

<sup>b</sup>% GDP

<sup>c</sup> $1 - G_w$

<sup>d</sup> $(\frac{\bar{w}}{\tau} (\frac{\bar{w}}{\tau \bar{w}})^{-\delta} - G_{wr})$

<sup>e</sup> $Var_t[c]^{0.5} \div \sigma_{1,t}$

Our first experiment is over a 20% discount rate for policy decisions compared to a 2% rate for welfare decisions. 20% is designed to approximate the political life span: discounting a fixed cash-flow at this rate would imply a Macaulay duration of 4.2 years — or that politicians' utility is as if they were receiving a zero-coupon bond that matures in 4.2 years, compared to nearly 50 years under the optimal discount rate. Table 3 shows that the welfare losses of this degree of political impatience are huge: citizens whose politicians operate with a discount rate of 20%, 18pp below the social rate of 2%, would pay 26% of the state's contribution to consumption in perpetuity to have a policymaker that discounts at the social rate! Just to give some idea of this magnitude, this is 650 times Pally and Robe (2003)'s estimates of the costs of removing macroeconomic volatility in the US and nearly 30 times larger than their mean estimates for African countries. The main reason is that the costs of volatility are here attributed to a fundamental characteristic (political impatience), which can also affect the ergodic mean level of utility.

But volatile asset returns — which as I argued could be due to domestic financial frictions — also imply large welfare costs. Table 3 shows that a rise in the standard deviation of asset returns by 0.4pp from the baseline has the same effect as politicians discounting at 20%. A 0.4pp rise means that the 95 % range for next-period returns broadens to 3% - 19 % from 4% - 17%.

Equivalent losses can only be generated by radically procyclical interest rates, such that a 1pp of GDP rise in non-discretionary revenue next period triggers a 0.6pp rise in the real return on net assets. In the absence of further evidence, I would judge this degree of procyclicality to be unlikely to be observed in practice; Kaminsky, Reinhart, and Végh (2004) estimated a significant procyclical linear dependence of the order of 0.27 (using the Treasury Bill rate for lower middle income countries).

The lowest rows of Table 3 present the notable result that it is impossible to generate a welfare loss of the same magnitude as implied by political impatience with greater revenue uncertainty. The largest compensation (of 11.1%) is for a increase in the standard deviation of resource revenues by 1.4pp. Beyond that, more volatile revenue can improve welfare by stimulating greater public saving. Thus even for those countries highly dependent on volatile resources, dependency need not be a problem per se. Rather the macroeconomic constraint is political impatience and the poor performance of its available instruments.<sup>29</sup>

Thus far, we have not been able to discriminate between political impatience and financial shallowness: both can potentially explain why natural resource abundance can lead to poor fiscal service. All we can confirm is that volatile resource returns by themselves are unlikely to be important. Can we discriminate further between political impatience and financial shallowness?

Table 3 compares the effect of a similar loss in welfare in terms of other more observable features. It shows that a greater impatience does generate startlingly different predictions than either more volatile asset returns or procyclicality, for the equivalent welfare loss. Most starkly of all, net assets will be lower — a whole 10pp of GDP lower — under political impatience. Under all of the other explanations, there are only at worst slightly more net assets held than the baseline. Considering that the level of net assets in Colombia was estimated to be well below the benchmark, it would seem that political impatience is the most likely channel through which resource dependence affects fiscal service, at least in Colombia.

Political impatience also makes discretionary expenditure exceptionally sensitive to resource returns. While in the baseline a 1pp rise in revenue leads to a 0.16 pp of GDP rise in expenditure, with political impatience this jumps to 0.26pp. Similarly, only because of impatience can we expect to see a much greater proportion of asset returns spent rather than absorbed back into saving: six times as much compared to the socially patient level! Finally political impatience predicts that discretionary expenditures will be half as volatile as the resource revenue. Under all other explanations, volatility remains at the baseline ratio of

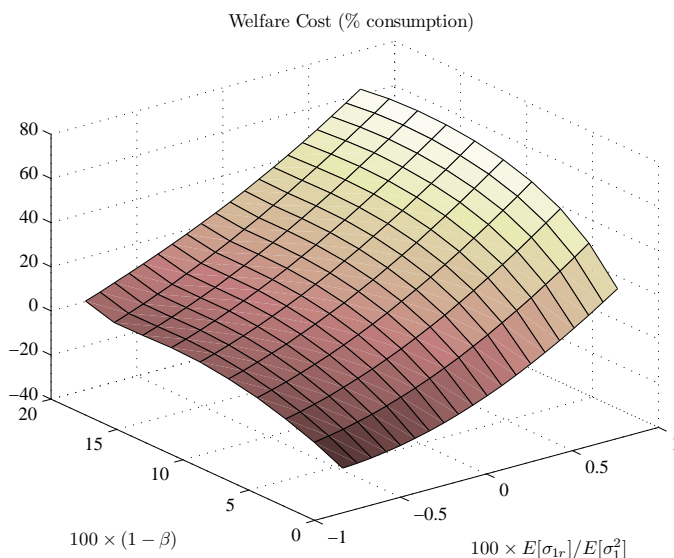
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<sup>29</sup>It might also matter that the level of revenue follows a log normal distribution which is skewed upwards. It is beyond the scope of this paper to analyse the tail risk of environmental damage from resource exploitation for example.

one fifth or thereabouts.

Can a countercyclical policy — even of moderate strength — offset the ill-effects of political impatience? In Chart 6, the compensating welfare measure is compared along two dimensions: countercyclicality and short-termism, allowing for interaction.

**Chart 6: Welfare effects of countercyclical rates and political short-termism<sup>a</sup>**




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<sup>a</sup>  $\frac{E[\sigma_{1r}]}{E[\sigma_1^2]}$  describes the linear dependence of real rates on non-discretionary revenues. The other axis describes the discount rate used for policy decisions relative to a 2% rate used for welfare evaluation.

Interestingly, it seems that only a fiercely aggressive countercyclical policy can offset the effects of extreme political impatience. A linear dependence of -0.8 between real returns on assets and surprises to non-discretionary revenue is far stronger than even the largest estimates obtained by Kaminsky, Reinhart, and Végh (2004) for developed countries. Countercyclicality of this degree would most probably interfere with other goals such as inflation and GDP stabilisation. The conclusion to draw is that political impatience may best have to be tackled at source rather than offset with a second-best countercyclical bias. Nevertheless, the experiment does warn that a combination of procyclical returns and political short-termism can have dire welfare consequences for the resource exporting economy: the surface slopes steeply upward towards the furthest corner to reach a mammoth loss of 60% of baseline consumption.

## 13 Conclusion and discussion

In this paper, I introduce two new elements into the analysis of the fiscal problem of volatile resources. Firstly, that revenues, expenditures and rates of returns on assets are uncertain all matter: they should be treated as known unknowns in the natural resource fiscal problem. Second that public capital should take its rightful place in the government's economic balance sheet, providing that we acknowledge that returns to public investment are of high risk and high social yield. The purpose is to discriminate between political impatience and a lack of financial depth as explanations of why some countries fail to successfully exploit the boon of natural resources. After calibrating and solving a model for Colombia, I found that political impatience as opposed to financial frictions is the most likely explanation. While both imply large welfare costs when the full effect of uncertainty is incorporated, only political impatience generates the plausible prediction that the holdings of effective net assets will be suboptimally small, and that current spending decisions are oversensitive to current revenues.

Thus the results of this paper justify a focus on institutions to remove policy impatience in the first instance for natural resource exporting countries rather than on alleviating financial frictions. Frankel (2010) offers some practical suggestions along these lines. In particular, he argues that forecasts for fiscal planning should be done independently of the political decision. I would only add that these forecasts should take account of uncertainty. My findings suggest that the consequences of underpredicting volatility can be more severe than optimism about growth.

It seems difficult to counteract this by other means. In principle making asset returns strongly countercyclical improves welfare and acts against short-termism. But the extent of this reaction has to be so extreme that it might well be at the cost of other policy goals. The best that can be done is to avoid policy settings which imply procyclical buffer returns when political institutions are weak. Thus there are costs to fixed exchange rate solutions and unsterilized foreign exchange interventions beyond the possibility that real wages are rigid.

Naturally, I have excluded other important aspects of the resource problem. One is the possibility of hedging through futures and other derivative contracts (Borensztein, Jeanne, and Sandri (2009)). Second that the private physical and human capital stock should also matter (Canuto and Cavallari (2012)). One hurdle to overcome in incorporating the private sector is to clarify how the government can influence private investment in these productive capitals. A third extension is to allow for data uncertainty, explicitly acknowledging that resource revenue forecasts are in effect noisy data about the future. In this way, we could tackle head on the problematics of separating forecasts of prices of the resource from those of volumes and, within each, separating cycle from trend.

# Appendices

## A The distribution of the exogenous processes

In this section I derive useful expressions based on the model for the exogenous processes: 18 and 19. Following Granger and Newbold (1976),

$$\mathbb{E}_s[\mathbf{x}_{s+1}] = \begin{pmatrix} e^{(1-\kappa_1)\bar{l}x_1 + \kappa_1 \ln(x_{1,s}) + \frac{d_{11}}{2}} \\ e^{(1-\kappa_2)\bar{l}x_2 + \kappa_2 \ln(x_{2,s}) + \frac{d_{22}}{2}} \\ e^{(1-\kappa_r)\bar{l}r + \kappa_r \ln(r_s) + \frac{d_{33}}{2}} \end{pmatrix}.$$

Similarly defining

$$\begin{pmatrix} \sigma_{1,s}^2 & \sigma_{12,s} & \sigma_{1r,s} \\ \sigma_{12,s} & \sigma_{2,s}^2 & \sigma_{2r,s} \\ \sigma_{1r,s} & \sigma_{2r,s} & \sigma_{r,s}^2 \end{pmatrix} \equiv \mathbb{E}_s[\mathbf{x}_{s+1}\mathbf{x}_{s+1}^T - \mathbb{E}[\mathbf{x}_{s+1}]\mathbb{E}[\mathbf{x}_{s+1}]^T],$$

the conditional covariances of the log-transformed process are

$$\begin{aligned} \sigma_{1,s}^2 &= e^{2(1-\kappa_1)\bar{l}x_1 + 2\kappa_1 \ln(x_{1,s}) + d_{11}} (e^{d_{11}} - 1) \\ \sigma_{2,s}^2 &= e^{2(1-\kappa_2)\bar{l}x_2 + 2\kappa_2 \ln(x_{2,s}) + d_{22}} (e^{d_{22}} - 1) \\ \sigma_{r,s}^2 &= e^{2(1-\kappa_r)\bar{l}r + 2\kappa_r \ln(r_s) + d_{33}} (e^{d_{33}} - 1) \\ \sigma_{12,s} &= e^{(1-\kappa_1)\bar{l}x_1 + \kappa_1 \ln(x_{1,s}) + \frac{d_{11}}{2} + (1-\kappa_2)\bar{l}x_2 + \kappa_2 \ln(x_{2,s}) + \frac{d_{22}}{2}} (e^{d_{12}} - 1) \\ \sigma_{1r,s} &= e^{(1-\kappa_1)\bar{l}x_1 + \kappa_1 \ln(x_{1,s}) + \frac{d_{11}}{2} + (1-\kappa_r)\bar{l}r + \kappa_r \ln(r_s) + \frac{d_{33}}{2}} (e^{d_{13}} - 1) \\ \sigma_{2r,s} &= e^{(1-\kappa_2)\bar{l}x_2 + \kappa_2 \ln(x_{2,s}) + \frac{d_{22}}{2} + (1-\kappa_r)\bar{l}r + \kappa_r \ln(r_s) + \frac{d_{33}}{2}} (e^{d_{23}} - 1). \end{aligned} \tag{48}$$

Their unconditional means are

$$\begin{aligned} \mathbb{E}[\sigma_{1,s}^2] &= e^{2\bar{l}x_1 + \frac{1+\kappa_1^2}{1-\kappa_1^2}d_{11}} (e^{d_{11}} - 1) \\ \mathbb{E}[\sigma_{2,s}^2] &= e^{2\bar{l}x_2 + \frac{1+\kappa_2^2}{1-\kappa_2^2}d_{22}} (e^{d_{22}} - 1) \\ \mathbb{E}[\sigma_{r,s}^2] &= e^{2\bar{l}r + \frac{1+\kappa_r^2}{1-\kappa_r^2}d_{33}} (e^{d_{33}} - 1) \\ \mathbb{E}[\sigma_{12,s}] &= e^{\bar{l}x_1 + \bar{l}x_2 + \frac{d_{11}}{2(1-\kappa_1^2)} + \frac{d_{22}}{2(1-\kappa_2^2)} + \frac{\kappa_1\kappa_2 d_{12}}{(1-\kappa_1\kappa_2)}} (e^{d_{12}} - 1) \\ \mathbb{E}[\sigma_{1r,s}] &= e^{\bar{l}x_1 + \bar{l}r + \frac{d_{11}}{2(1-\kappa_1^2)} + \frac{d_{33}}{2(1-\kappa_r^2)} + \frac{\kappa_1\kappa_r d_{13}}{(1-\kappa_1\kappa_r)}} (e^{d_{13}} - 1) \\ \mathbb{E}[\sigma_{2r,s}] &= e^{\bar{l}x_2 + \bar{l}r + \frac{d_{22}}{2(1-\kappa_2^2)} + \frac{d_{33}}{2(1-\kappa_r^2)} + \frac{\kappa_2\kappa_r d_{23}}{(1-\kappa_2\kappa_r)}} (e^{d_{23}} - 1). \end{aligned} \tag{49}$$

## B Derivation of solution

Differentiating  $f_1(\cdot)$  from 34 with respect to  $c_{t+1}$  and  $r_{t+1}$ , we have

$$\frac{\partial^2 f_1}{\partial c_{t+1}^2} = \frac{\gamma(\gamma+1)(1-\delta)}{c_{t+1}^2} (\beta(\frac{c_{t+1}\tau}{c_t})^{-\gamma} r_{t+1}) (\frac{w_t}{\tau\tilde{w}})^{-\delta}, \quad (50)$$

$$\frac{\partial^2 f_1}{\partial r_{t+1}^2} = 0, \quad (51)$$

and

$$\frac{\partial^2 f_1}{\partial c_{t+1} \partial r_{t+1}} = \frac{d^2 f_1}{dr_{t+1} dc_{t+1}} = \frac{-\gamma(1-\delta)}{c_{t+1} r_{t+1}} (\beta(\frac{c_{t+1}\tau}{c_t})^{-\gamma} r_{t+1}) (\frac{w_t}{\tau\tilde{w}})^{-\delta}. \quad (52)$$

Following our general formulation of section 2, we apply equation 8 to 34 using expressions 50, 51 and 52. This leads to equation 37 presented in the main text.

$$\begin{aligned} \mathbb{E}_t [f_1(c_{t+1}, r_{t+1}, c_t, w_t)] &\approx \Phi(\mathbb{E}_t[c_{t+1}, r_{t+1}, c_t, w_t]) \\ &\equiv \beta(1-\delta) (\frac{\mathbb{E}_t[c_{t+1}]\tau}{c_t})^{-\gamma} \mathbb{E}_t[r_{t+1}] (\frac{w_t}{\tau\tilde{w}})^{-\delta} - 1 \\ &+ \beta(1-\delta) (\frac{\mathbb{E}_t[c_{t+1}]\tau}{c_t})^{-\gamma} \mathbb{E}_t[r_{t+1}] (\frac{w_t}{\tau\tilde{w}})^{-\delta} \frac{1}{2} \frac{\gamma(\gamma+1)}{\mathbb{E}_t[c_{t+1}]^2} Var_t[c_{t+1}] \\ &- \beta(1-\delta) (\frac{\mathbb{E}_t[c_{t+1}]\tau}{c_t})^{-\gamma} \mathbb{E}_t[r_{t+1}] (\frac{w_t}{\tau\tilde{w}})^{-\delta} \frac{\gamma}{\mathbb{E}_t[c_{t+1}] \mathbb{E}_t[r_{t+1}]} Cov_t[c_{t+1}, r_{t+1}] \end{aligned} \quad (53)$$

$$\begin{aligned} \Rightarrow \hat{\Phi} &\equiv 1 - (\frac{\mathbb{E}_t[c_{t+1}]\tau}{c_t})^\gamma \frac{1}{\beta(1-\delta)\mathbb{E}_t[r_{t+1}]} (\frac{w_t}{\tau\tilde{w}})^\delta \\ &+ \frac{\gamma(\gamma+1)}{2} \frac{Var_t[c_{t+1}]}{\mathbb{E}_t[c_{t+1}]^2} \\ &- \gamma \frac{Cov_t[c_{t+1}, r_{t+1}]}{\mathbb{E}_t[c_{t+1}] \mathbb{E}_t[r_{t+1}]} \approx 0. \end{aligned} \quad (54)$$

In going from 53 to 54 we are assuming that the expression  $\beta(1-\delta) (\frac{\mathbb{E}_t[c_{t+1}]\tau}{c_t})^{-\gamma} \mathbb{E}_t[r_{t+1}] (\frac{w_t}{\tau\tilde{w}})^{-\delta}$  is always non-zero. Pushing the expectations rule 41 one period forward,

$$w_{t+1} = \bar{w} + G_{ww}(w_t - \bar{w}) + G_{wr}(r_{t+1} - \bar{r}) + G_{w1}(x_{1,t+1} - \bar{x}_1) + G_{w2}(x_{2,t+1} - \bar{x}_2), \quad (55)$$

Taking expectations at time  $t$  of the budget constraint 16 (in model units) one period forward,

$$\mathbb{E}_t c_{t+1} = \mathbb{E}_t x_{1,t+1} - \mathbb{E}_t x_{2,t+1} + (\frac{w_t}{\tau})(\frac{w_t}{\tau\tilde{w}})^{-\delta} \mathbb{E}_t r_{t+1} - \mathbb{E}_t w_{t+1}. \quad (56)$$

Substituting in from equation 55, we have

$$\begin{aligned} \mathbb{E}_t [c_{t+1}] &= \mathbb{E}_t [x_{1,t+1}] - \mathbb{E}_t [x_{2,t+1}] \\ &+ (G_{ww} - 1)\bar{w} + G_{wr}\bar{r} + G_{w1}\bar{x}_1 + G_{w2}\bar{x}_2 \\ &+ (\frac{w_t}{\tau})(\frac{w_t}{\tau\tilde{w}})^{-\delta} - G_{wr}\mathbb{E}_t[r_{t+1}] - G_{ww}w_t - G_{w1}\mathbb{E}_t[x_{1,t+1}] - G_{w2}\mathbb{E}_t[x_{2,t+1}]. \end{aligned} \quad (57)$$

It follows that

$$\begin{aligned}
Var_t [c_{t+1}] &= (1 - G_{w1})^2 Var_t [x_{1,t+1}] + (1 + G_{w2})^2 Var_t [x_{2,t+1}] \\
&+ \left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right)^2 Var_t [r_{t+1}] \\
&- 2(1 - G_{w1})(1 + G_{w2}) Cov_t [x_{1,t+1}, x_{2,t+1}] \\
&+ 2(1 - G_{w1}) \left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right) Cov_t [x_{1,t+1}, r_{t+1}] \\
&- 2(1 + G_{w2}) \left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right) Cov_t [x_{2,t+1}, r_{t+1}] \\
&= (1 - G_{w1})^2 \sigma_{1,t}^2 + (1 + G_{w2})^2 \sigma_{2,t}^2 + \left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right)^2 \sigma_{r,t}^2 \\
&- 2(1 - G_{w1})(1 + G_{w2}) \sigma_{12,t} + 2(1 - G_{w1}) \left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right) \sigma_{1r,t} \\
&- 2(1 + G_{w2}) \left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right) \sigma_{2r,t}
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
Cov_t [c_{t+1}, r_{t+1}] &= \left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right) Var_t [r_{t+1}] \\
&+ (1 - G_{w1}) Cov_t [x_{1,t+1}, r_{t+1}] - (1 + G_{w2}) Cov_t [x_{2,t+1}, r_{t+1}] \\
&= \left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right) \sigma_{r,t}^2 + (1 - G_{w1}) \sigma_{1r,t} - (1 + G_{w2}) \sigma_{2r,t}.
\end{aligned} \tag{59}$$

Combining expressions 54, 58 and 59,

$$\begin{aligned}
\hat{\Phi}(\mathbb{E}_t [c_{t+1}], \mathbb{E}_t [r_{t+1}], c_t, w_t) &= 1 - \left( \frac{\mathbb{E}_t [c_{t+1}] \tau}{c_t} \right)^\gamma \frac{1}{\beta(1 - \delta) \mathbb{E}_t [r_{t+1}]} \left( \frac{w_t}{\tau \tilde{w}} \right)^\delta \\
&+ \frac{\gamma(\gamma + 1)}{2} \frac{(1 - G_{w1})^2 \sigma_{1,t}^2 + (1 + G_{w2})^2 \sigma_{2,t}^2}{\mathbb{E}_t [c_{t+1}]^2} \\
&+ \frac{\gamma(\gamma + 1)}{2} \frac{\left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right)^2 \sigma_{r,t}^2}{\mathbb{E}_t [c_{t+1}]^2} \\
&- \gamma(\gamma + 1) \frac{(1 - G_{w1})(1 + G_{w2}) \sigma_{12,t} - (1 - G_{w1}) \left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right) \sigma_{1r,t}}{\mathbb{E}_t [c_{t+1}]^2} \\
&- \gamma(\gamma + 1) \frac{(1 + G_{w2}) \left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right) \sigma_{2r,t}}{\mathbb{E}_t [c_{t+1}]^2} \\
&- \gamma \frac{\left( \left( \frac{w_t}{\tau} \right) \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} - G_{wr} \right) \sigma_{r,t}^2}{\mathbb{E}_t [c_{t+1}] \mathbb{E}_t [r_{t+1}]} \\
&- \gamma \frac{(1 - G_{w1}) \sigma_{1r,t}}{\mathbb{E}_t [c_{t+1}] \mathbb{E}_t [r_{t+1}]} \\
&+ \gamma \frac{(1 + G_{w2}) \sigma_{2r,t}}{\mathbb{E}_t [c_{t+1}] \mathbb{E}_t [r_{t+1}]}.
\end{aligned} \tag{60}$$



We now derive the following useful partial derivatives of equation 60 with respect to all variables (including variances)

$$\begin{aligned}
\frac{\partial \hat{\Phi}}{\partial w_t} &= \delta \left( \frac{\mathbb{E}_t [c_{t+1}] \tau}{c_t} \right)^\gamma \frac{1}{\beta(1-\delta) \mathbb{E}_t [r_{t+1}] w_t} \left( \frac{w_t}{\tau \tilde{w}} \right)^\delta \\
&\quad + \frac{\gamma(\gamma+1)(1-\delta) \left( \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right) \sigma_{r,t}^2}{\tau \mathbb{E}_t [c_{t+1}]^2} \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} \\
&\quad + \frac{\gamma(\gamma+1)(1-\delta) (1-G_{w1}) \sigma_{1r,t} \left( \frac{w_t}{\tilde{w}} \right)^{-\delta} - (1+G_{w2}) \sigma_{2r,t}}{\tau \mathbb{E}_t [c_{t+1}]^2} \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta} \\
&\quad - \frac{\gamma(1-\delta) \sigma_{r,t}^2}{\tau \mathbb{E}_t [c_{t+1}] \mathbb{E}_t [r_{t+1}]} \left( \frac{w_t}{\tau \tilde{w}} \right)^{-\delta}, \tag{61}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \hat{\Phi}}{\partial \mathbb{E}_t [r_{t+1}]} &= \left( \frac{\mathbb{E}_t [c_{t+1}] \tau}{c_t} \right)^\gamma \frac{1}{\beta(1-\delta) \mathbb{E}_t [r_{t+1}]^2} \left( \frac{w_t}{\tau \tilde{w}} \right)^\delta \\
&\quad + \gamma \frac{\left( \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right) \sigma_{r,t}^2}{\mathbb{E}_t [c_{t+1}] \mathbb{E}_t [r_{t+1}]^2} \\
&\quad + \gamma \frac{(1-G_{w1}) \sigma_{1r,t} - (1+G_{w2}) \sigma_{2r,t}}{\mathbb{E}_t [c_{t+1}] \mathbb{E}_t [r_{t+1}]^2}, \tag{62}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \hat{\Phi}}{\partial \mathbb{E}_t [c_{t+1}]} &= -\gamma \left( \frac{\mathbb{E}_t [c_{t+1}] \tau}{c_t} \right)^\gamma \frac{1}{\beta(1-\delta) \mathbb{E}_t [c_{t+1}] \mathbb{E}_t [r_{t+1}]} \left( \frac{w_t}{\tau \tilde{w}} \right)^\delta \\
&\quad - \gamma(\gamma+1) \frac{(1-G_{w1})^2 \sigma_{1,t}^2 + (1+G_{w2})^2 \sigma_{2,t}^2 + \left( \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right)^2 \sigma_{r,t}^2}{\mathbb{E}_t [c_{t+1}]^3} \\
&\quad + 2\gamma(\gamma+1) \frac{(1-G_{w1})(1+G_{w2}) \sigma_{12,t} - (1-G_{w1}) \left( \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right) \sigma_{1r,t}}{\mathbb{E}_t [c_{t+1}]^3} \\
&\quad + 2\gamma(\gamma+1) \frac{(1+G_{w2}) \left( \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right) \sigma_{2r,t}}{\mathbb{E}_t [c_{t+1}]^3} \\
&\quad + \gamma \frac{\left( \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right) \sigma_{r,t}^2}{\mathbb{E}_t [c_{t+1}]^2 \mathbb{E}_t [r_{t+1}]} \\
&\quad + \gamma \frac{(1-G_{w1}) \sigma_{1r,t} - (1+G_{w2}) \sigma_{2r,t}}{\mathbb{E}_t [c_{t+1}]^2 \mathbb{E}_t [r_{t+1}]}, \tag{63}
\end{aligned}$$

$$\frac{\partial \hat{\Phi}}{\partial \mathbb{E}_t [c_t]} = \gamma \left( \frac{\mathbb{E}_t [c_{t+1}] \tau}{c_t} \right)^\gamma \frac{1}{\beta(1-\delta) c_t \mathbb{E}_t [r_{t+1}]} \left( \frac{w_t}{\tau \tilde{w}} \right)^\delta, \tag{64}$$

and

$$\frac{\partial \hat{\Phi}}{\partial \sigma_{1,t}^2} = \frac{\gamma(\gamma+1) (1-G_{w1})^2}{2 \mathbb{E}_t [c_{t+1}]^2}, \tag{65}$$

$$\frac{\partial \hat{\Phi}}{\partial \sigma_{2,t}^2} = \frac{\gamma(\gamma+1)(1+G_{w2})^2}{2 \mathbb{E}_t[c_{t+1}]^2}, \quad (66)$$

$$\frac{\partial \hat{\Phi}}{\partial \sigma_{r,t}^2} = \frac{\gamma(\gamma+1) \left( \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right)^2}{2 \mathbb{E}_t[c_{t+1}]^2} - \gamma \frac{\left( \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right)}{\mathbb{E}_t[c_{t+1}] \mathbb{E}_t[r_{t+1}]}, \quad (67)$$

$$\frac{\partial \hat{\Phi}}{\partial \sigma_{12,t}} = -\gamma(\gamma+1) \frac{(1-G_{w1})(1+G_{w2})}{\mathbb{E}_t[c_{t+1}]^2}, \quad (68)$$

$$\frac{\partial \hat{\Phi}}{\partial \sigma_{1r,t}} = +\gamma(\gamma+1) \frac{(1-G_{w1}) \left( \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right)}{\mathbb{E}_t[c_{t+1}]^2} - \gamma \frac{(1-G_{w1})}{\mathbb{E}_t[c_{t+1}] \mathbb{E}_t[r_{t+1}]}, \quad (69)$$

and

$$\frac{\partial \hat{\Phi}}{\partial \sigma_{2r,t}} = -\gamma(\gamma+1) \frac{(1+G_{w2}) \left( \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right)}{\mathbb{E}_t[c_{t+1}]^2} + \gamma \frac{(1+G_{w2})}{\mathbb{E}_t[c_{t+1}] \mathbb{E}_t[r_{t+1}]}. \quad (70)$$

Next we differentiate equation 57 in terms of all its variables:

$$\frac{\partial \mathbb{E}_t[c_{t+1}]}{\partial \mathbb{E}_t[r_{t+1}]} = \left( \frac{w_t}{\tau} \right) \left( \frac{\bar{w}}{\tilde{w}} \right)^{-\delta} - G_{wr}, \quad (71)$$

$$\frac{\partial \mathbb{E}_t[c_{t+1}]}{\partial \mathbb{E}_t[x_{1,t+1}]} = 1 - G_{w1}, \quad (72)$$

$$\frac{\partial \mathbb{E}_t[c_{t+1}]}{\partial \mathbb{E}_t[x_{2,t+1}]} = -(1+G_{w2}), \quad (73)$$

and

$$\frac{\partial \mathbb{E}_t[c_{t+1}]}{\partial w_t} = (1-\delta) \frac{\mathbb{E}_t[r_{t+1}]}{w_t} \left( \frac{w_t}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{ww}. \quad (74)$$

From the exogenous processes A and 48 we also have that

$$\frac{\partial \mathbb{E}_t[r_{t+1}]}{\partial r_t} = \frac{\kappa_r}{r_t} \mathbb{E}_t[r_{t+1}], \quad (75)$$

$$\frac{\partial \mathbb{E}_t[x_{i,t+1}]}{\partial x_{i,t}} = \frac{\kappa_i}{x_{i,t}} \mathbb{E}_t[x_{i,t+1}], \quad (76)$$

$$\frac{\partial \sigma_{1,t}^2}{\partial x_{1,t}} = \sigma_{1,t}^2 \frac{2\kappa_1}{x_{1,t}}, \quad (77)$$

$$\frac{\partial \sigma_{2,t}^2}{\partial x_{2,t}} = \sigma_{2,t}^2 \frac{2\kappa_2}{x_{2,t}}, \quad (78)$$

$$\frac{\partial \sigma_{r,t}^2}{\partial r_t} = \sigma_{r,t}^2 \frac{2\kappa_r}{r_t}, \quad (79)$$

$$\frac{\partial \sigma_{12,t}}{\partial x_{1,t}} = \sigma_{12,t} \frac{\kappa_1}{x_{1,t}}, \quad (80)$$

$$\frac{\partial \sigma_{12,t}}{\partial x_{2,t}} = \sigma_{12,t} \frac{\kappa_2}{x_{2,t}}, \quad (81)$$

$$\frac{\partial \sigma_{1r,t}}{\partial x_{1,t}} = \sigma_{1r,t} \frac{\kappa_1}{x_{1,t}}, \quad (82)$$

$$\frac{\partial \sigma_{1r,t}}{\partial r_t} = \sigma_{1r,t} \frac{\kappa_r}{r_t}, \quad (83)$$

$$\frac{\partial \sigma_{2r,t}}{\partial x_{2,t}} = \sigma_{2r,t} \frac{\kappa_2}{x_{2,t}}, \quad (84)$$

and

$$\frac{\partial \sigma_{2r,t}}{\partial r_t} = \sigma_{2r,t} \frac{\kappa_r}{r_t}. \quad (85)$$

We differentiate the budget constraint 16 after having substituted out for  $w_t$  using the expectations rule 41 with respect to the states:

$$\frac{\partial c_t}{\partial r_t} = \left(\frac{w_{t-1}}{\tau}\right) \left(\frac{w_{t-1}}{\tilde{w}}\right)^{-\delta} - G_{wr}, \quad (86)$$

$$\frac{\partial c_t}{\partial x_{1,t}} = 1 - G_{w1}, \quad (87)$$

$$\frac{\partial c_t}{\partial x_{2,t}} = -(1 + G_{w2}), \quad (88)$$

$$\frac{\partial c_t}{\partial w_{t-1}} = (1 - \delta) \frac{r_t}{w_{t-1}} \left(\frac{w_{t-1}}{\tau}\right)^{1-\delta} \tilde{w}^\delta - G_{ww}, \quad (89)$$

From the expectations rule 41,

$$\frac{\partial w_t}{\partial r_t} = G_{wr}, \quad (90)$$

$$\frac{\partial w_t}{\partial x_{i,t}} = G_{wi}, \quad (91)$$

and

$$\frac{\partial w_t}{\partial w_{t-1}} = G_{ww}. \quad (92)$$

The total derivatives of  $\hat{\Phi}$  with respect to the states are then:

$$\begin{aligned} \frac{d\hat{\Phi}}{dw_{t-1}} &= \frac{\partial \hat{\Phi}}{\partial w_t} \frac{\partial w_t}{\partial w_{t-1}} + \frac{\partial \hat{\Phi}}{\partial \mathbb{E}_t[c_{t+1}]} \frac{\partial \mathbb{E}_t[c_{t+1}]}{\partial w_t} \frac{\partial w_t}{\partial w_{t-1}} \\ &+ \frac{\partial \hat{\Phi}}{\partial c_t} \frac{\partial c_t}{\partial w_{t-1}} \end{aligned} \quad (93)$$

$$\begin{aligned}
\frac{d\hat{\Phi}}{dr_t} &= \frac{\partial\hat{\Phi}}{\partial w_t} \frac{\partial w_t}{\partial r_t} + \frac{\partial\hat{\Phi}}{\partial \mathbb{E}_t[c_{t+1}]} \frac{\partial \mathbb{E}_t[c_{t+1}]}{\partial w_t} \frac{\partial w_t}{\partial r_t} \\
&+ \frac{\partial\hat{\Phi}}{\partial c_t} \frac{\partial c_t}{\partial r_t} \\
&+ \frac{\partial\hat{\Phi}}{\partial \mathbb{E}_t[r_{t+1}]} \frac{\partial \mathbb{E}_t[r_{t+1}]}{\partial r_t} + \frac{\partial\hat{\Phi}}{\partial \mathbb{E}_t[c_{t+1}]} \frac{\partial \mathbb{E}_t[c_{t+1}]}{\partial \mathbb{E}_t[r_{t+1}]} \frac{\partial \mathbb{E}_t[r_{t+1}]}{\partial r_t} \\
&+ \frac{\partial\hat{\Phi}}{\partial \sigma_{r,t}^2} \frac{\partial \sigma_{r,t}^2}{\partial r_t} + \frac{\partial\hat{\Phi}}{\partial \sigma_{1r,t}} \frac{\partial \sigma_{1r,t}}{\partial r_t} + \frac{\partial\hat{\Phi}}{\partial \sigma_{2r,t}} \frac{\partial \sigma_{2r,t}}{\partial r_t}
\end{aligned} \tag{94}$$

and

$$\begin{aligned}
\frac{d\hat{\Phi}}{dx_{i,t}} &= \frac{\partial\hat{\Phi}}{\partial w_t} \frac{\partial w_t}{\partial x_{i,t}} + \frac{\partial\hat{\Phi}}{\partial \mathbb{E}_t[c_{t+1}]} \frac{\partial \mathbb{E}_t[c_{t+1}]}{\partial w_t} \frac{\partial w_t}{\partial x_{i,t}} \\
&+ \frac{\partial\hat{\Phi}}{\partial c_t} \frac{\partial c_t}{\partial x_{i,t}} \\
&+ \frac{\partial\hat{\Phi}}{\partial \mathbb{E}_t[c_{t+1}]} \frac{\partial \mathbb{E}_t[c_{t+1}]}{\partial \mathbb{E}_t[y_{t+1}]} \frac{\partial \mathbb{E}_t[y_{t+1}]}{\partial x_{i,t}} \\
&+ \frac{\partial\hat{\Phi}}{\partial \sigma_{i,t}^2} \frac{\partial \sigma_{i,t}^2}{\partial x_{i,t}} + \frac{\partial\hat{\Phi}}{\partial \sigma_{ir,t}} \frac{\partial \sigma_{ir,t}}{\partial x_{i,t}} + \frac{\partial\hat{\Phi}}{\partial \sigma_{12,t}} \frac{\partial \sigma_{12,t}}{\partial x_{i,t}}
\end{aligned} \tag{95}$$

The budget constraint 16 at the risky steady state becomes

$$\bar{c} = \bar{x}_1 - \bar{x}_2 + \bar{r} \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} - \bar{w}. \tag{96}$$

Substituting 17 into 60 and evaluating at the risky steady state,

$$\begin{aligned}
\hat{\Phi}(\bar{w}, \bar{r}, \bar{x}_i) &= 1 - \frac{\tau^\gamma}{\beta(1-\delta)\bar{r}} \left( \frac{\bar{w}}{\tau\bar{w}} \right)^\delta \\
&+ \frac{\gamma(\gamma+1)}{2} \frac{(1-G_{w1})^2 \mathbb{E}[\sigma_{1,s}^2] + (1+G_{w2})^2 \mathbb{E}[\sigma_{2,s}^2]}{(\bar{x}_1 - \bar{x}_2 + \bar{r} \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} \tilde{w}^\delta - \bar{w})^2} \\
&+ \frac{\gamma(\gamma+1)}{2} \frac{\left( \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right)^2 \mathbb{E}[\sigma_{r,s}^2]}{(\bar{x}_1 - \bar{x}_2 + \bar{r} \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} - \bar{w})^2} \\
&- \gamma(\gamma+1) \frac{(1-G_{w1})(1+G_{w2}) \mathbb{E}[\sigma_{12,s}^2] - (1-G_{w1}) \left( \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right) \mathbb{E}[\sigma_{1r,s}^2]}{(\bar{x}_1 - \bar{x}_2 + \bar{r} \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} \tilde{w}^\delta - \bar{w})^2} \\
&- \gamma(\gamma+1) \frac{(1+G_{w2}) \left( \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right) \mathbb{E}[\sigma_{2r,s}^2]}{(\bar{x}_1 - \bar{x}_2 + \bar{r} \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} \tilde{w}^\delta - \bar{w})^2} \\
&- \gamma \frac{\left( \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} \tilde{w}^\delta - G_{wr} \right) \mathbb{E}[\sigma_{r,s}^2]}{\bar{r}(\bar{x}_1 - \bar{x}_2 + \bar{r} \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} \tilde{w}^\delta - \bar{w})} \\
&- \gamma \frac{(1-G_{w1}) \mathbb{E}[\sigma_{1r,s}^2] - (1+G_{w2}) \mathbb{E}[\sigma_{2r,s}^2]}{\bar{r}(\bar{x}_1 - \bar{x}_2 + \bar{r} \left( \frac{\bar{w}}{\tau} \right)^{1-\delta} \tilde{w}^\delta - \bar{w})} \\
&\approx 0.
\end{aligned} \tag{97}$$

The risky steady state are the values of  $G_{ww}$ ,  $G_{wr}$ ,  $G_{wi}$ ,  $\bar{w}$  and  $\bar{c}$  that constitute the joint solution of equations 93, 94 and 95 at the steady state, as well as 96 and 97. The solutions are a function of the underlying parameters  $\beta$ ,  $\tau$ ,  $\delta$  and  $\gamma$  and the parameters describing the exogenous processes —  $\bar{l}x_1$ ,  $\bar{l}x_2$ ,  $\kappa_1$ ,  $\kappa_2$ ,  $\bar{l}r$ ,  $\kappa_r$ ,  $\mathbf{D}$  — whose baseline values are given in the following table

**Table 4: Calibrations**

Mean values:	$\ln(N - DR)$ $\bar{l}x_1 = 0.98$	$\ln(N - DE)$ $\bar{l}x_2 = 0.80$	$\ln(RR)$ $\bar{l}r = 0.1$
AR(1) params.:	$\ln(N - DR)$ $\kappa_1 = 0.07$	$\ln(N - DE)$ $\kappa_2 = -0.04$	$\ln(RR)$ $\kappa_r = 0.90$
Covariances:	$\ln(N - DR)$ $d_{11} = 0.06^2$	$\ln(N - DE)$ -	$\ln(RR)$ -
	$\log(N-DE)$ $d_{12} = 0$	$d_{22} = 0.06^2$	-
	$\ln(RR)$ $d_{13} = 0$	$d_{23} = 0$	$d_{33} = 0.0077^2$

## C Approximation to the conditional variance of future consumption

We work with the following linearised version of the state system and the budget constraint

$$\begin{aligned}
w_{s+1} &= \bar{w} + G_{ww}(w_s - \bar{w}) + G_{wr}(r_{s+1} - \bar{r}) + G_{w1}(x_{1,s+1} - \bar{x}_1) + G_{w2}(x_{2,s+1} - \bar{x}_2); \\
x_{1,s+1} &\approx (1 - \kappa_1)\bar{x}_1 + \kappa_1 x_{1,t} + \bar{x}_1 u_{1,s+1}; \\
x_{2,s+1} &\approx (1 - \kappa_2)\bar{x}_2 + \kappa_2 x_{2,t} + \bar{x}_2 u_{2,s+1}; \\
r_{s+1} &\approx (1 - \kappa_r)\bar{r} + \kappa_r r_s + \bar{r} u_{s+1}; \\
\hat{c}_{s+1} &\approx ((1 - \delta)\frac{\bar{r}}{\tau}(\frac{\bar{w}}{\tilde{w}\tau})^{-\delta} - G_{ww})w_s + ((\frac{\bar{w}}{\tau})^{1-\delta}\tilde{w}^\delta - G_{wr})r_{s+1} + (1 - G_{w1})x_{1,s+1} - (1 + G_{w2})x_{2,s+1} \\
&\quad - (\frac{\bar{w}}{\tau})^{1-\delta}\tilde{w}^\delta\bar{r} + (G_{ww} + 1)\bar{w} + G_{wr}\bar{r} + G_{w1}\bar{x}_1 + G_{w2}\bar{x}_2.
\end{aligned} \tag{98}$$

Define  $\mathbf{z}_n \equiv (w_n, x_{1,n}, x_{2,n}, r_n)^T$ . Then

$$\mathbf{z}_{n+1} = \Omega_1^{n+1-t}\mathbf{z}_t + \sum_{k=1}^{n+1-t} \Omega_1^{n+1-t-k}\Omega_2\mathbf{D}^{\frac{1}{2}}\mathbf{v}_{t+1} + \Omega_3 \tag{99}$$

for  $n \geq t$ . Here

$$\Omega_1 \equiv \begin{pmatrix} G_{ww} & G_{w1}\kappa_1 & G_{w2}\kappa_2 & G_{wr}\kappa_r \\ 0 & \kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 \\ 0 & 0 & 0 & \kappa_r \end{pmatrix},$$

$$\Omega_2 \equiv \begin{pmatrix} G_{w1}\bar{x}_1 & G_{w2}\bar{x}_2 & G_{wr}\bar{r} \\ \bar{x}_1 & 0 & 0 \\ 0 & \bar{x}_2 & 0 \\ 0 & 0 & \bar{r} \end{pmatrix},$$

$\mathbf{D}^{\frac{1}{2}}$  is the Cholesky decomposition of the matrix  $\mathbf{D}$ , defined in equation 19,

$$\mathbf{\Omega}_3 \equiv \begin{pmatrix} (1 - G_{ww})\bar{w} - \kappa_r G_{wr}\bar{r} - \kappa_1 G_{w1}\bar{x}_1 - \kappa_2 G_{w2}\bar{x}_2 \\ (1 - \kappa_1)\bar{x}_1 \\ (1 - \kappa_2)\bar{x}_2 \\ (1 - \kappa_r)\bar{r} \end{pmatrix}$$

and  $\mathbf{v}_{t+1}$  is a vector of three mean zero, unit variance, independent normally distributed shocks, and  $\mathbf{I}_N$ , a  $N \times N$  identity matrix. Using equation 99 we can calculate  $E_t[\mathbf{z}_{s+1}^2 - E_t[\mathbf{z}_{s+1}]\mathbf{z}_s^T - E_t[\mathbf{z}_{s+1}]E_t[\mathbf{z}_s]]$  for all  $s > t$ . The terms in these matrices give us the necessary expressions to calculate  $Var_t[\hat{c}_s] = E_t[\hat{c}_{s+1}^2 - E_t[\hat{c}_{s+1}]^2]$ , using the linear approximation to the budget constraint in the last row of equation 99. Our approximation to utility follows from inserting these terms  $Var_t[\hat{c}_s]$ ,  $s > t$  into the expression ?? in the main text.

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